Collisions and Spirals

Joint with J. Lind & S. Rohde

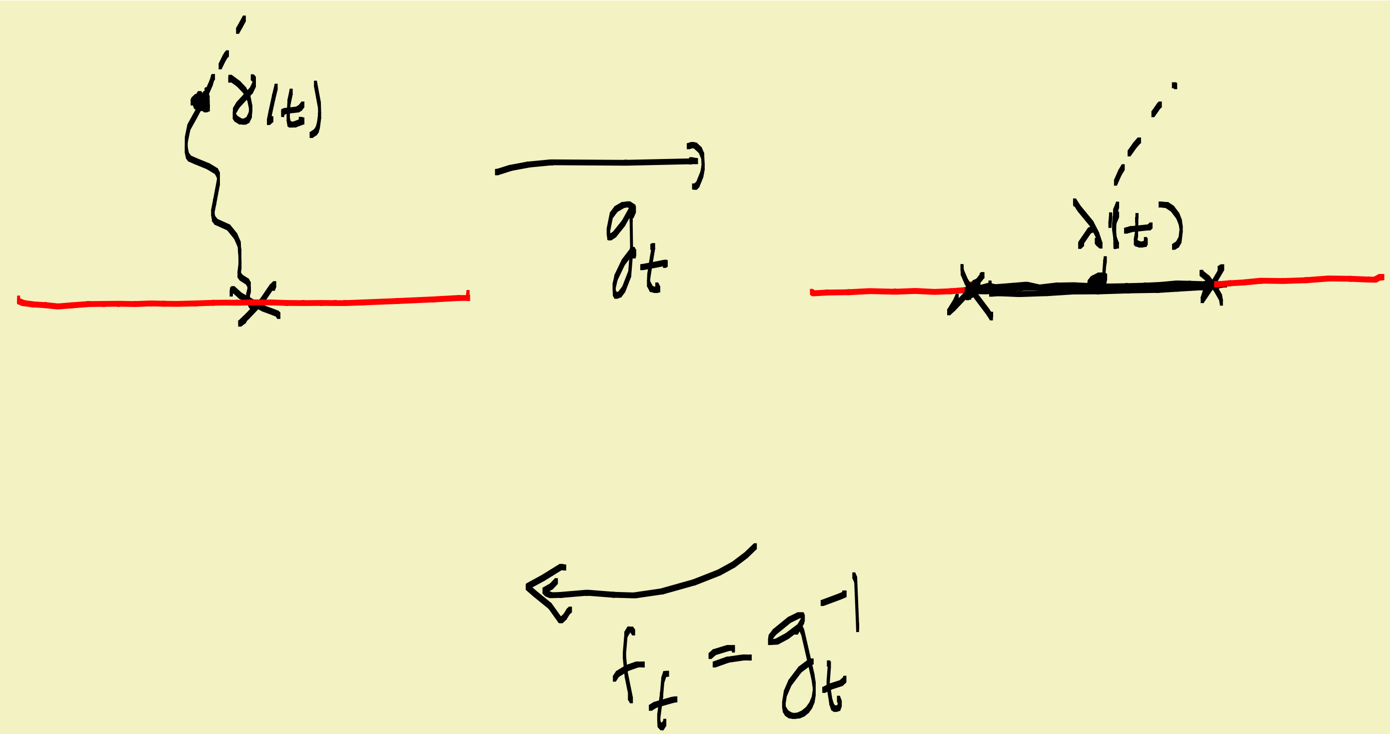
Rome Sept. 2008

- WHAT CAUSES THE LOEWNER TRACE TO HIT THE BOUNDARY IN FINITE TIME?

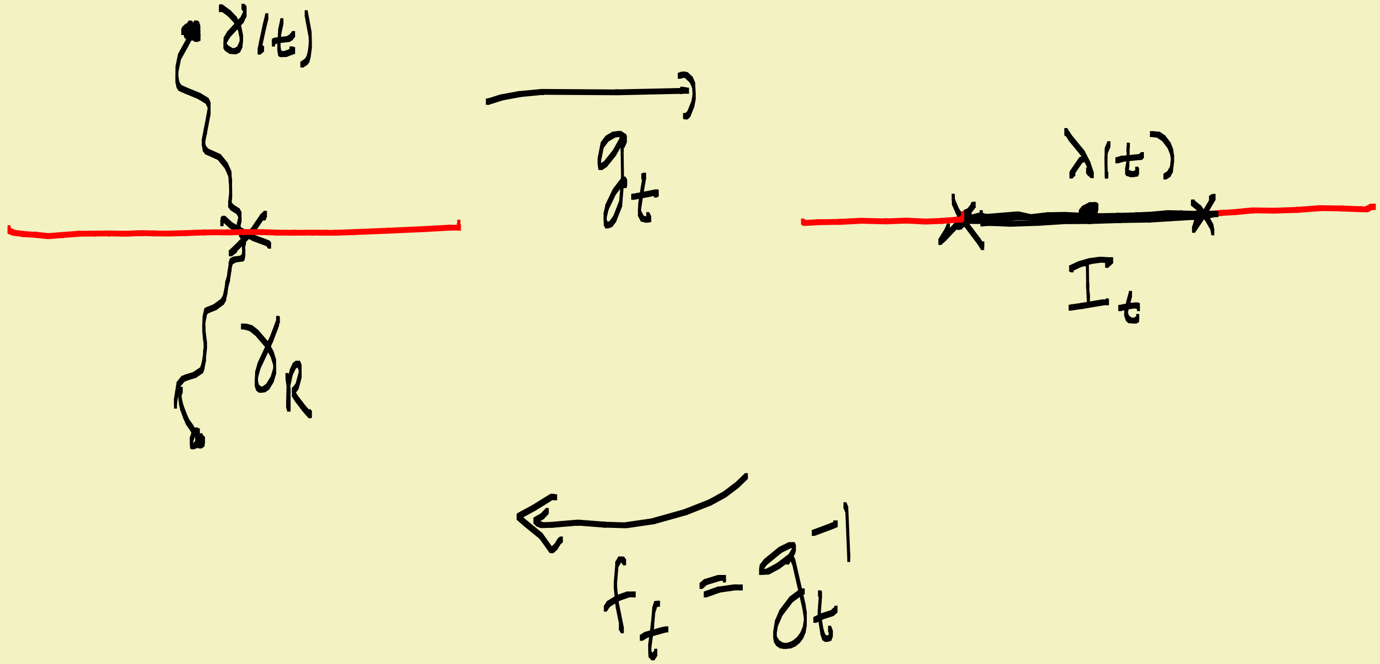
- (OMER ANGEL): IF THE TRACE FOR λ IS CONT. ON $[0,1]$, IS THE TRACE FOR $r\lambda$ CONT. $(r<1)$?

$$\dot{g} = \frac{2}{g-\lambda} \qquad g = g_t(z) \qquad g_0(z) \equiv z$$

DRIVING TERM



Fix t: g_t, f_t conformal maps

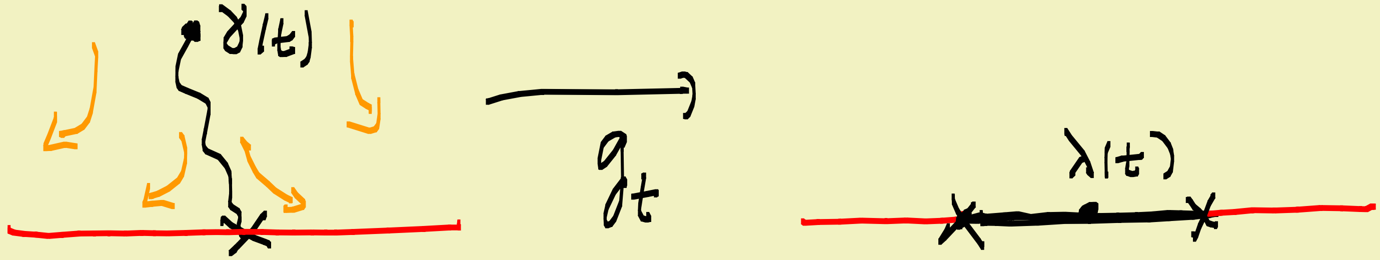


REFLECT ABOUT $\mathbb{R}$:

$$g(z) = z + 2t/z + O\left(\frac{1}{z^2}\right) \text{ near } \infty$$

$K_t \equiv \mathbb{H} \setminus f_t(\mathbb{H}) \qquad t \equiv \text{hcap}(K_t)$

"PARAM. BY H CAP"

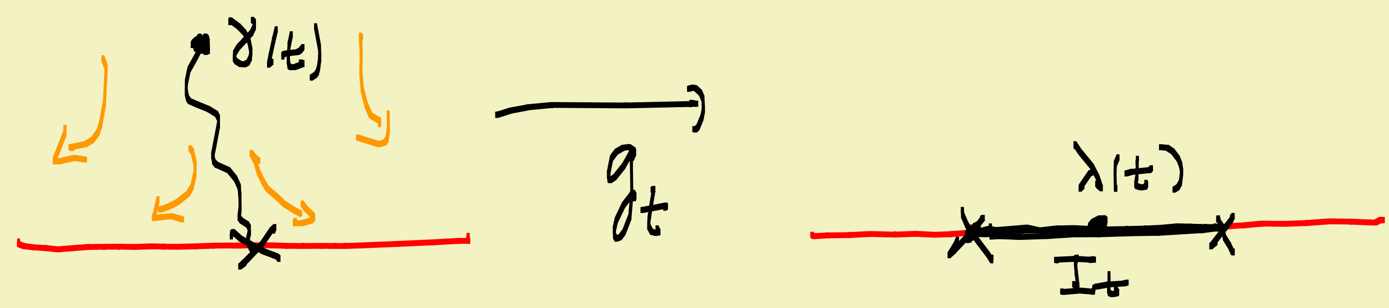


fix z, g_t is a flow

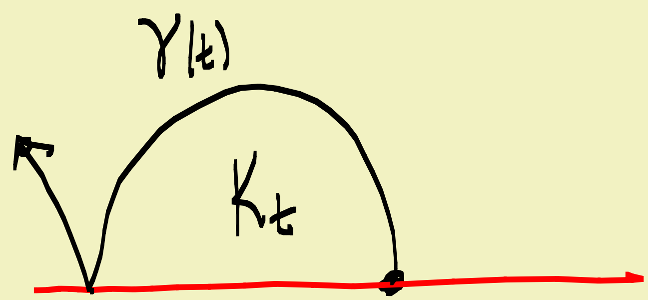
$$f_t = g_t^{-1}$$

HULL: K_t = POINTS SWALLOWED AT TIME $\le t$ BY $\mathbb{R}$

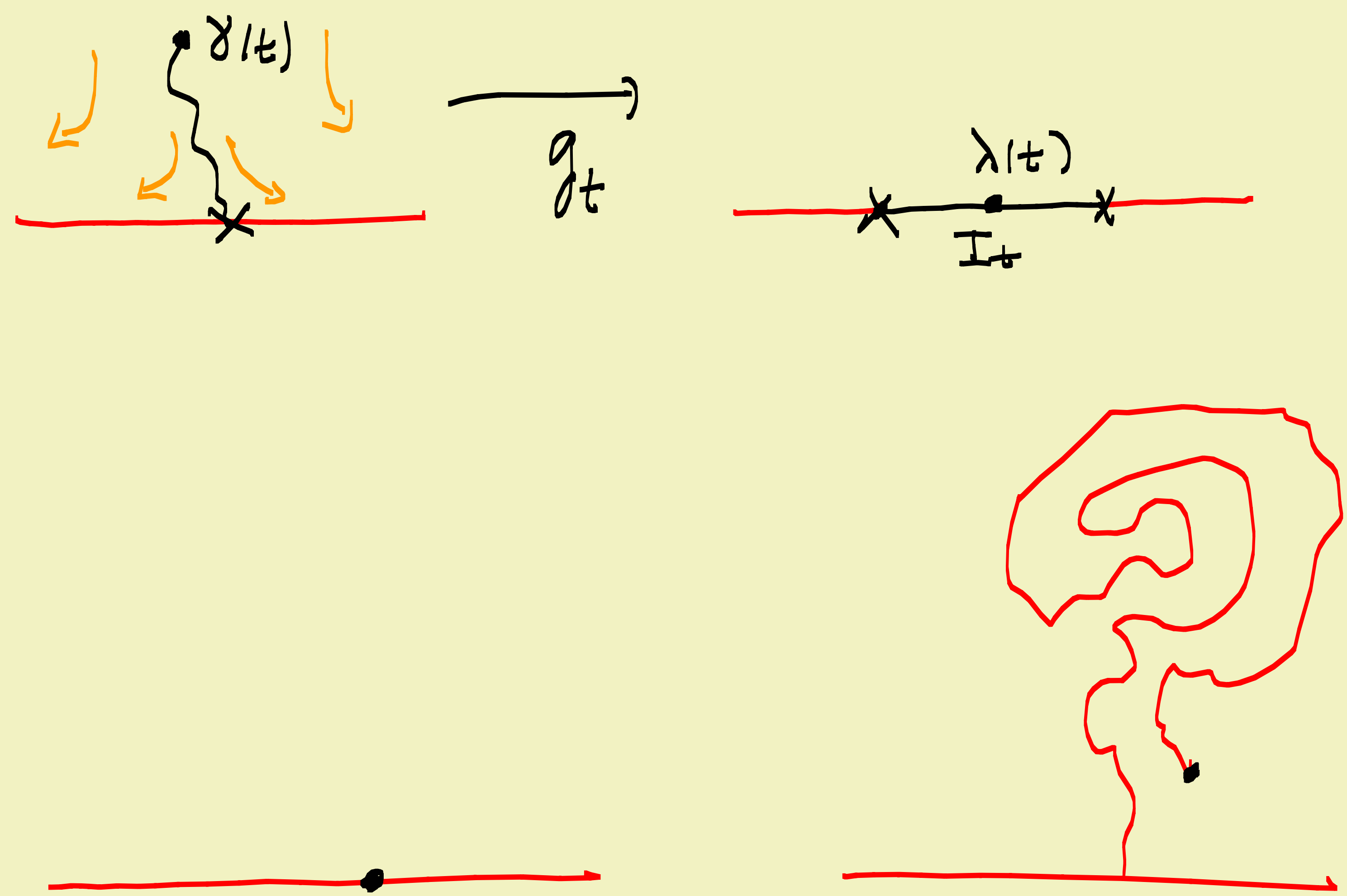
TRACE: $\gamma(t) = \bigcup_{u \le t} \partial K_u$



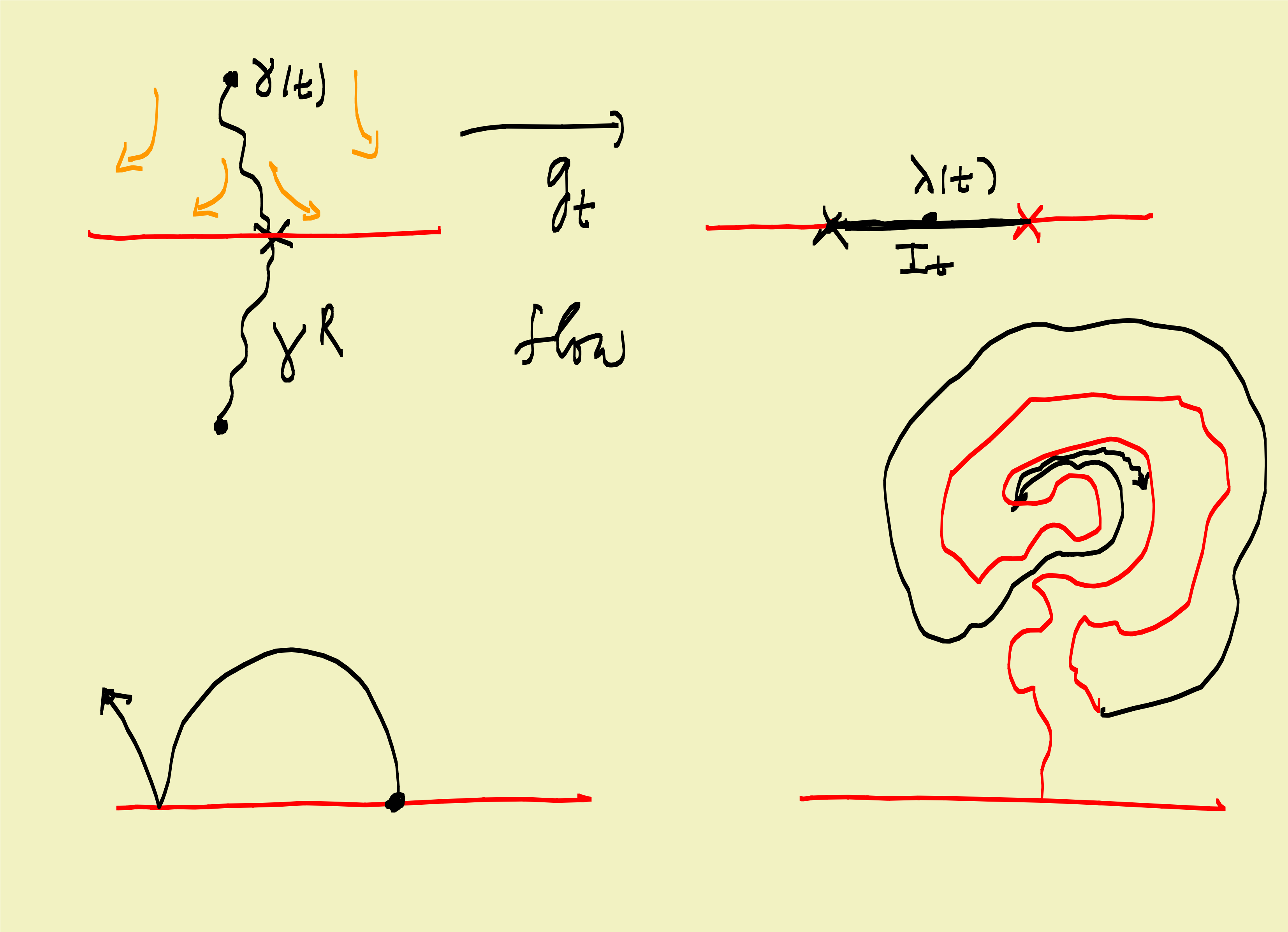
$\gamma(t)$
g_t
$\lambda(t)$
I_t



$\gamma(t)$
K_t



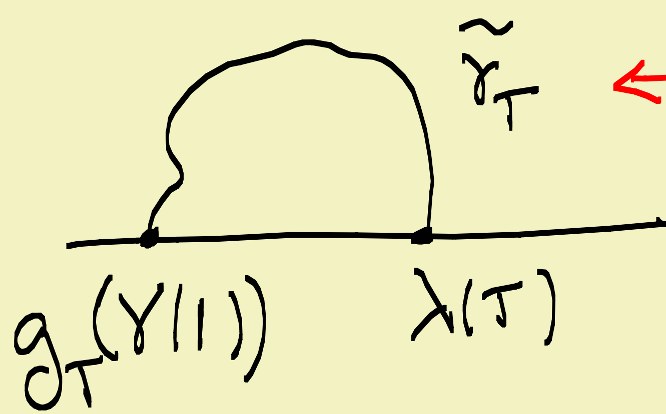
γ(t)
g_t
λ(t)
I_t


γ(t)
γ^R
g_t
flow
λ(t)
I_t

HOW $\gamma(t_1)$ MEETS $\gamma(t_0)$ DEPENDS ON BOTH $\lambda(t)$ NEAR t_1 AND NEAR t_0

(WLOG $t_1 = 1$)

SO WE APPLY g_T, $T < 1$, T NEAR 1.



DRIVEN BY $\lambda(T+t)$

$0 \le t \le 1-T$

RENORMALIZATION:

$$\gamma_T(t) \equiv \frac{g_T(\gamma(T+t(1-T)))}{\sqrt{1-T}} \qquad 0 \le t \le 1$$

$$hcap(\gamma_T) = 1$$

DRIVEN BY $\lambda_T(t) = \dfrac{\lambda(T+t(1-T))}{\sqrt{1-T}}$

REPARAMETRIZE (TIME CHANGE)

WANT: $s(t) - s(T) = s\left(\dfrac{t-T}{1-T}\right)$

$$s'(t) = s'(0)/(1-t)$$

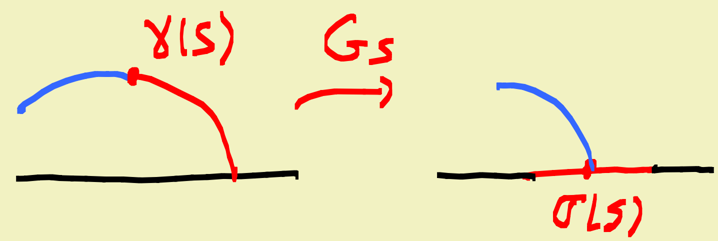
SO: $s(t) = \log \dfrac{1}{1-t} \qquad t = 1 - e^{-s}$

$$G_s = \frac{g_t}{\sqrt{1-t}} \qquad 0 \le s < \infty$$

$$\dot{G}_s \equiv \frac{dG}{ds} = \frac{2}{G-\sigma} + \frac{G}{2}$$

$$\sigma(s) = \frac{\lambda(t)}{\sqrt{1-t}} \qquad \text{(DRIVING TERM)}$$

$$\sigma_S(s) \equiv \sigma(S+s)$$



$$\gamma_S \equiv G_S\left(\gamma[S,\infty)\right)$$ — DRIVEN BY σ_S

$$G^{\sigma}_{s+S} = G^{\sigma_S}_{s} \circ G^{\sigma}_{S}$$

INVARIANT CURVES ?

<u>PROP.</u> $\gamma = \gamma_s \quad \forall\, s \in [0,1]$

$\iff \sigma(u) \equiv K$ (CONSTANT)

$\iff \lambda(t) = K\sqrt{1-t}$

PF: $\sigma(s+u) = \sigma_s(u) = \sigma(u) \quad \forall\, u$

$\therefore\ \sigma$ CONST.

KAGER, NIENHUIS, KADANOFF

$$\lambda(t) = K\sqrt{1-t}$$

$$s = \log 1/1-t$$

$$\dot{G} = \frac{2}{G-K} + \frac{G}{2}$$

SEPARATION OF VARIABLES

OBTAINED <u>IMPLICIT</u> EQ. FOR G (AND g_t)

MORE GEOMETRIC VIA INVARIANT CURVES:

KAGER, NIENHUIS, KADANOFF

$$\lambda(t) = K\sqrt{1-t}$$

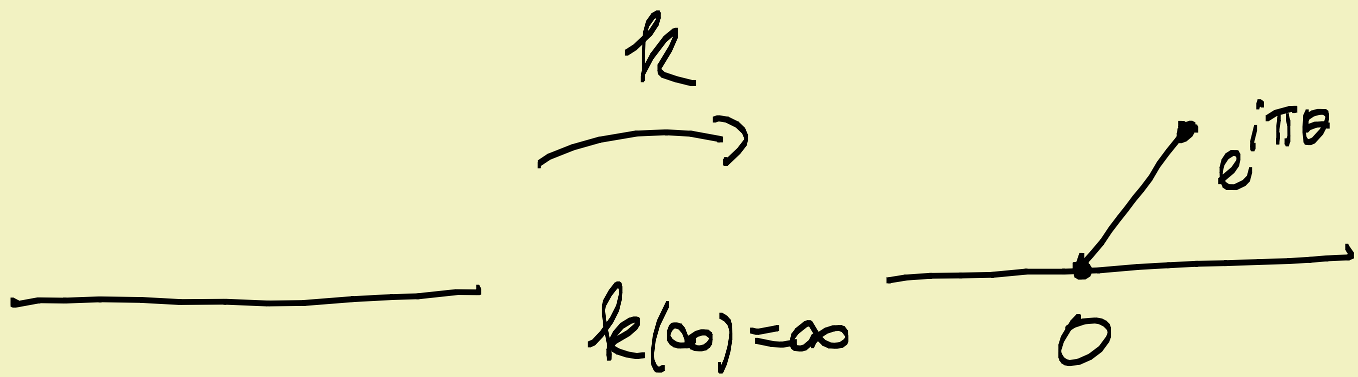
$$s = \log 1/1-t$$

$$\dot{G} = \frac{2}{G-K} + \frac{G}{2}$$

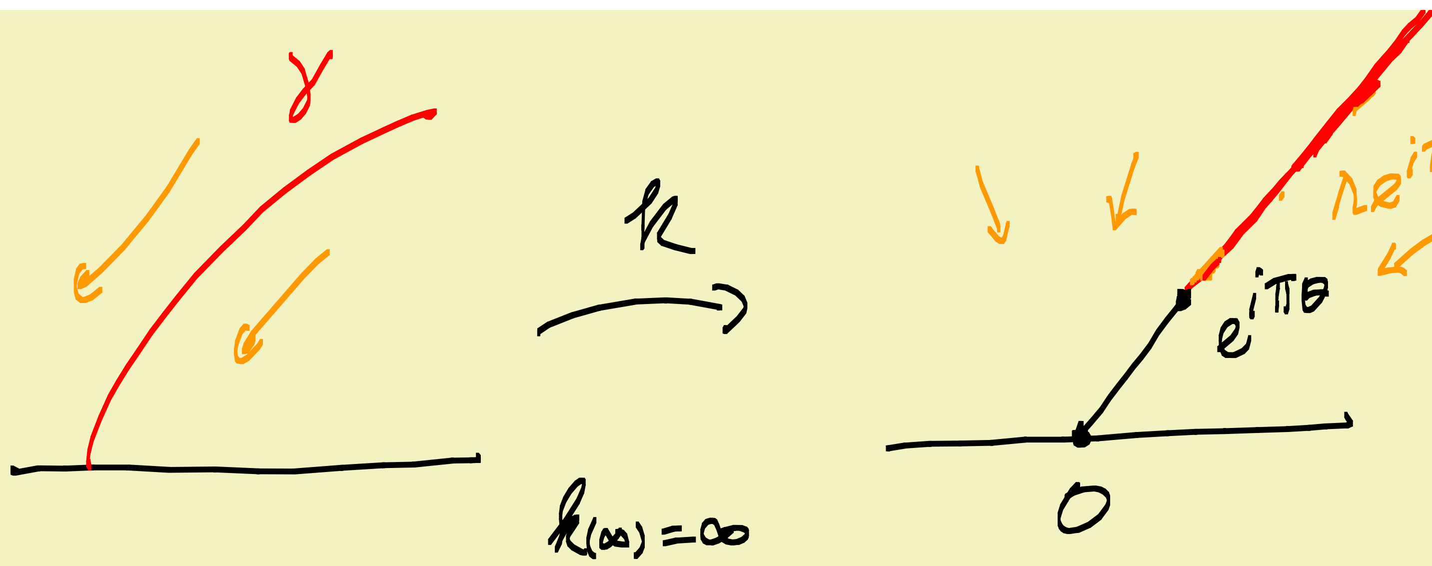
SEPARATION OF VARIABLES

OBTAINED <u>IMPLICIT</u> EQ. FOR G (AND g_t)

MORE GEOMETRIC VIA INVARIANT CURVES:

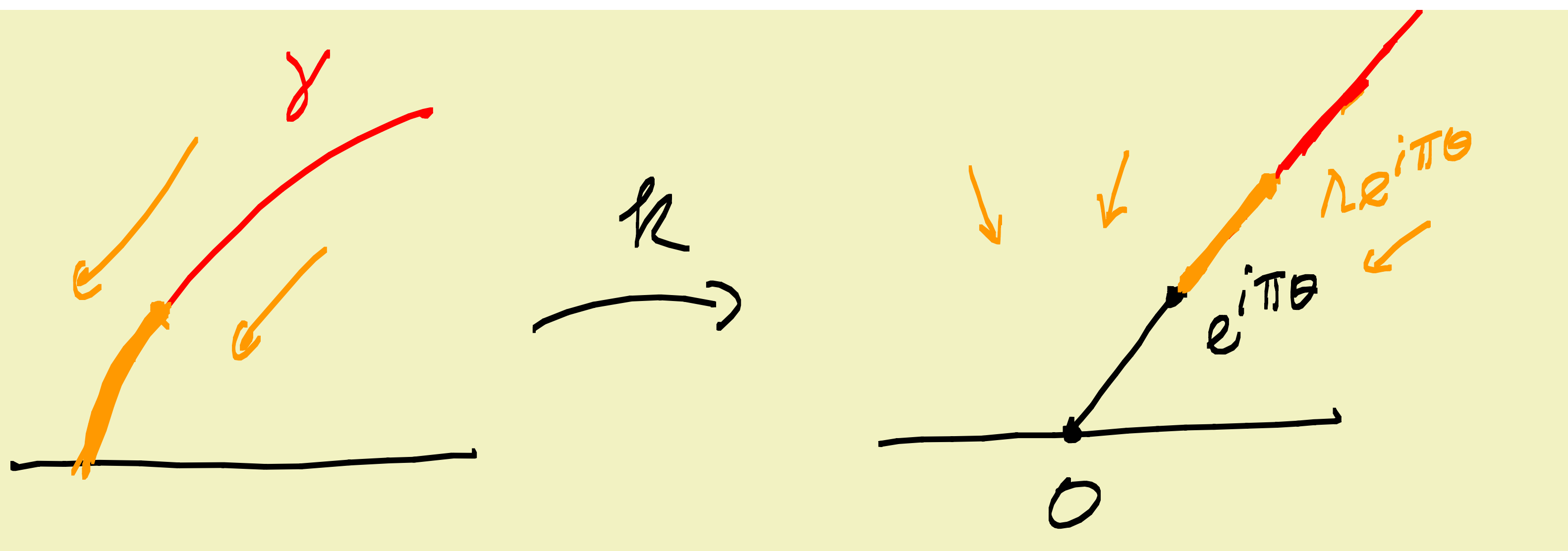


building block for "zipper"



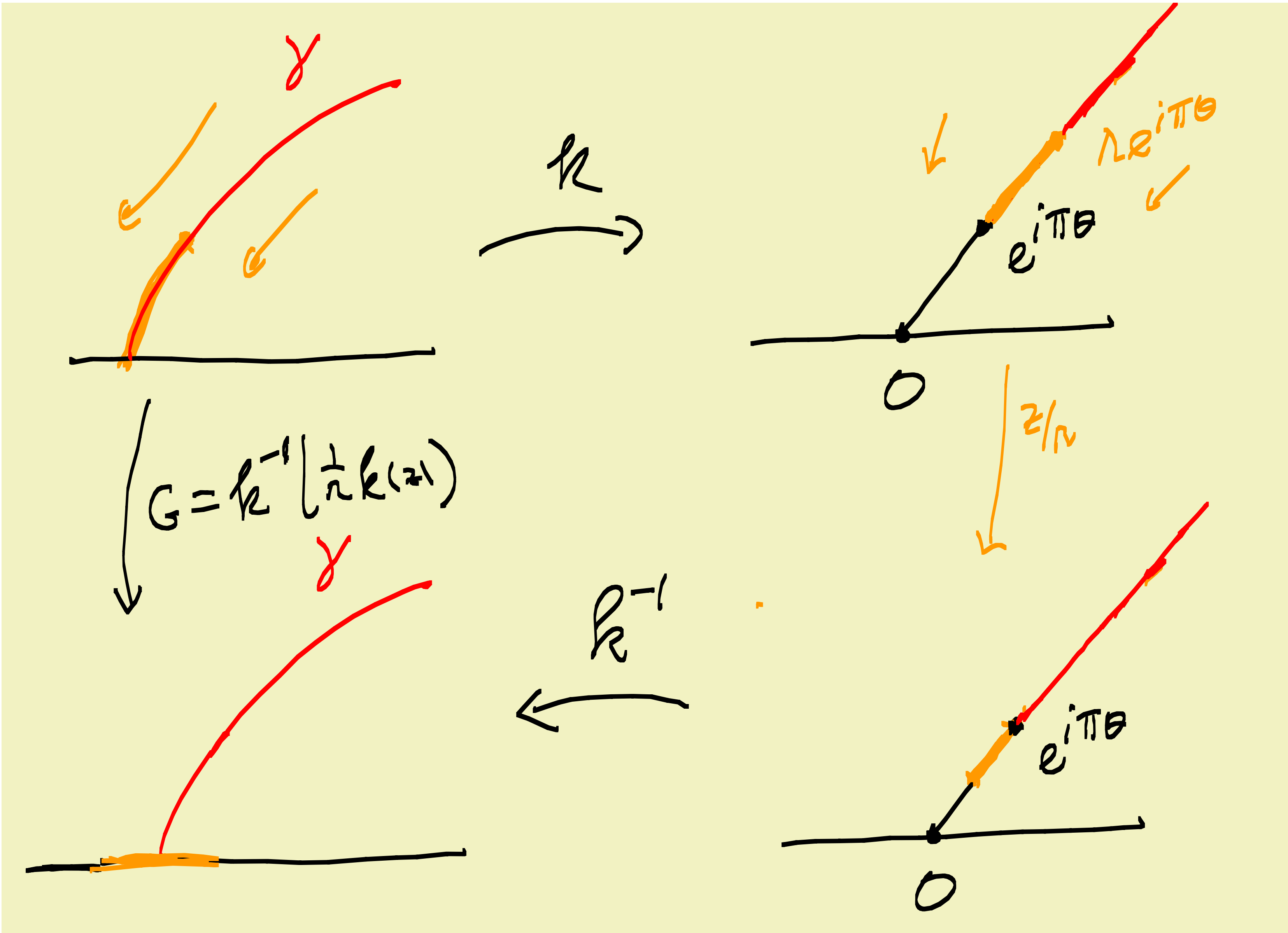
flow $z \to z/r$ $(r > 1)$

"Captured" points: $te^{i\pi\theta}$: $1 \le t \le r$



flow $z \to z/r$

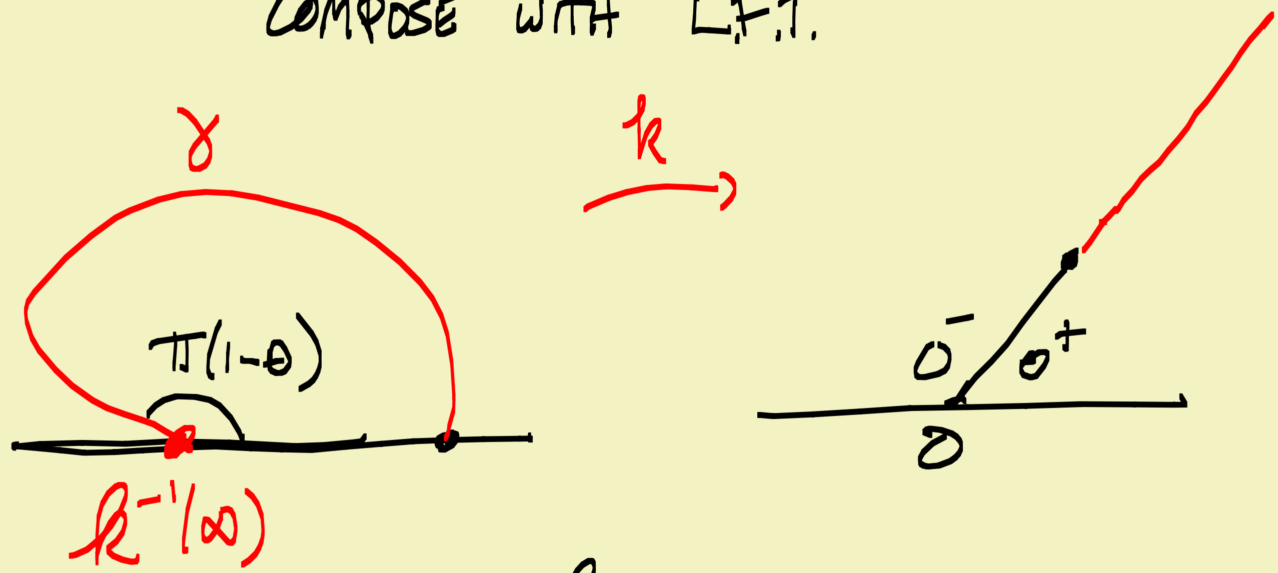
"Captured" points: $te^{i\pi\theta}$: $1 \leq t \leq r$



γ
k
$e^{i\pi\theta}$
$\lambda e^{i\pi\theta}$
0
z/λ
$G = k^{-1}\left(\frac{1}{\lambda}k(z)\right)$
k^{-1}

TO INTERSECT IN <u>FINITE</u> TIME,

COMPOSE WITH L.F.T.

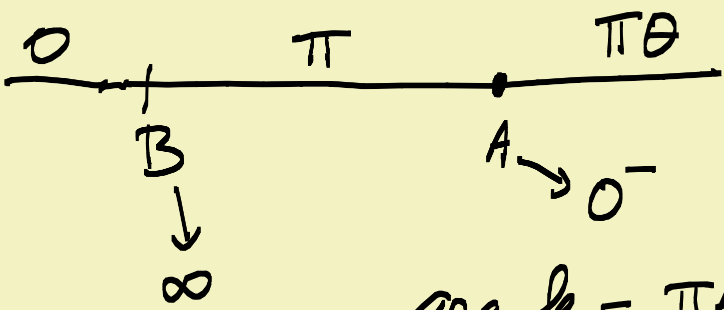


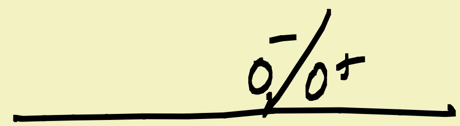
$$k(\infty) = \theta^+$$

$$G_\lambda = k^{-1}\left(\frac{1}{\lambda} k(z)\right)$$

By prop $\lambda(t) = K\sqrt{1-t}$

$\arg k$:





$$\arg k = \pi\theta + (1-\theta)\arg(z-A) - \arg(z-B)$$

$$k = c\, e^{i\pi\theta} \frac{(z-A)^{1-\theta}}{z-B}$$

plug into $\dot{G} = \frac{2}{G-K} + G/2$

$$\sigma(\theta) \equiv K = 2\left(\sqrt{1-\theta} + 1/\sqrt{1-\theta}\right)$$

$$\lambda(t) = K\sqrt{1-t}$$

$$g_t(z) = \sqrt{1-t}\, k^{-1}\left((1-t)^{-\frac{\theta}{2}} k(z)\right)$$

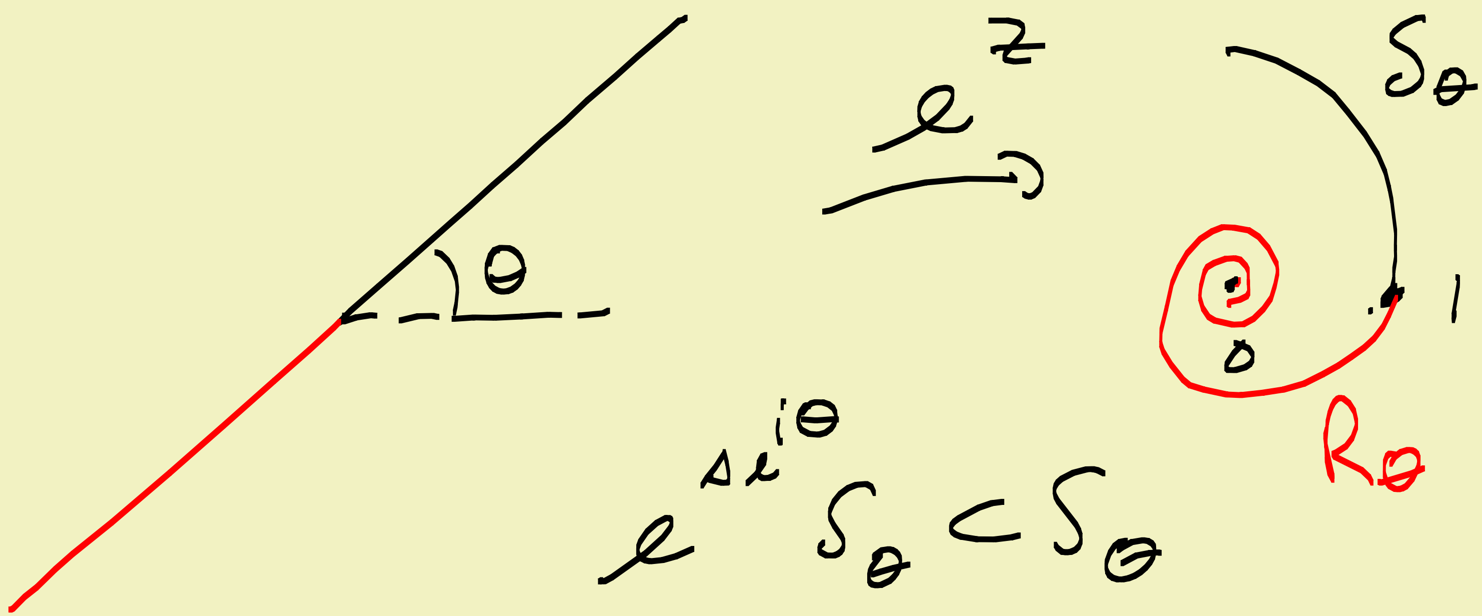
$0 < \theta < 1 \iff 4 < K < \infty$

use θ^- instead of θ^+

GET $-\infty < K < -4$

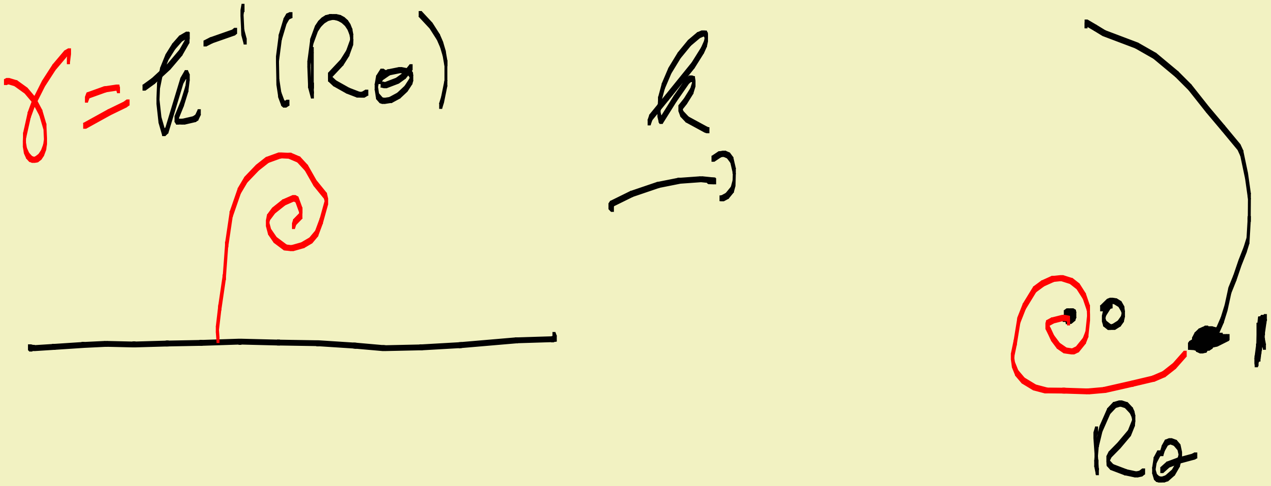
ANOTHER INVARIANT CURVE:

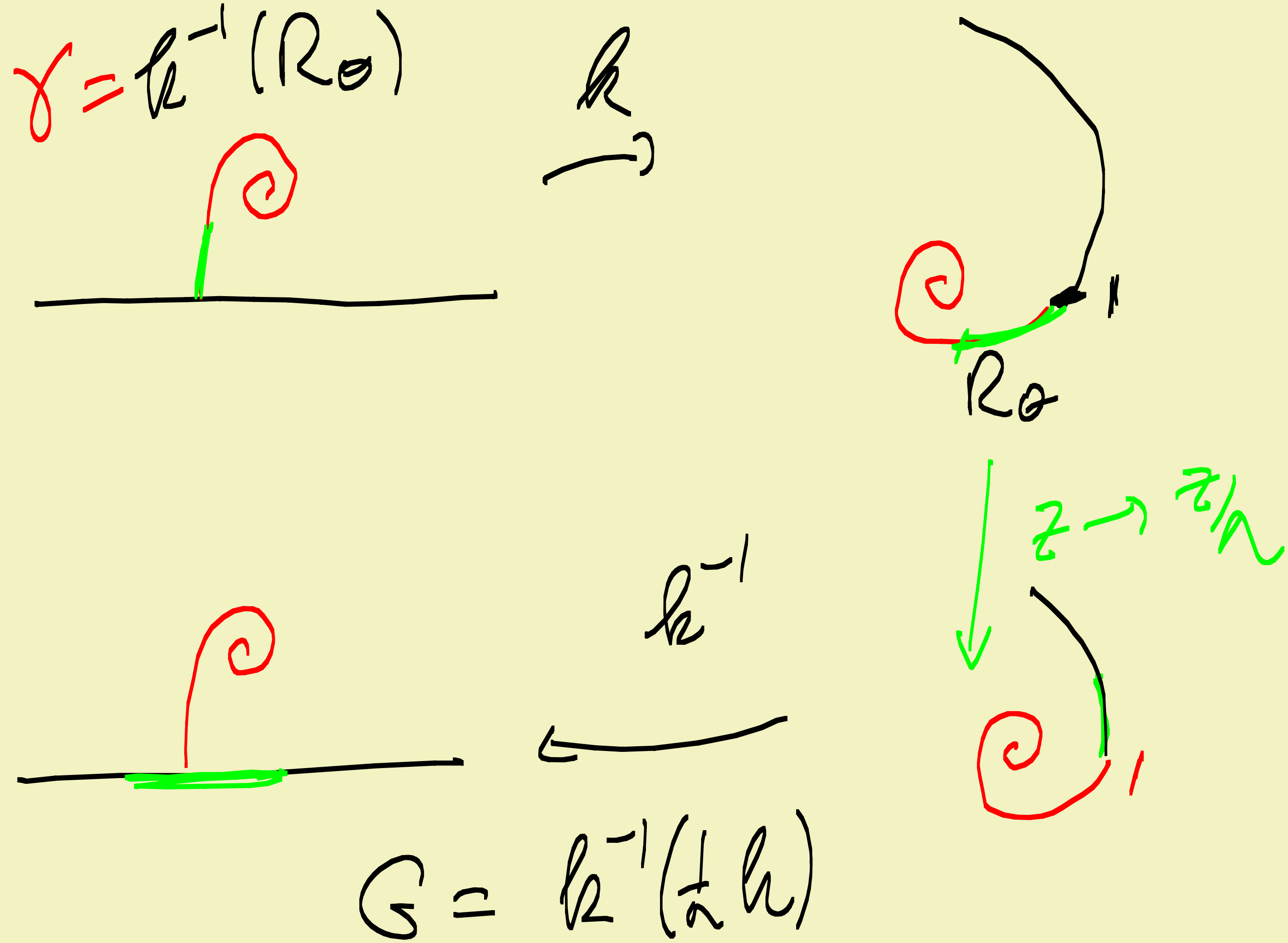
LOGARITHMIC SPIRAL



$$e^{\Delta e^{i\theta}} S_\theta \subset S_\theta$$

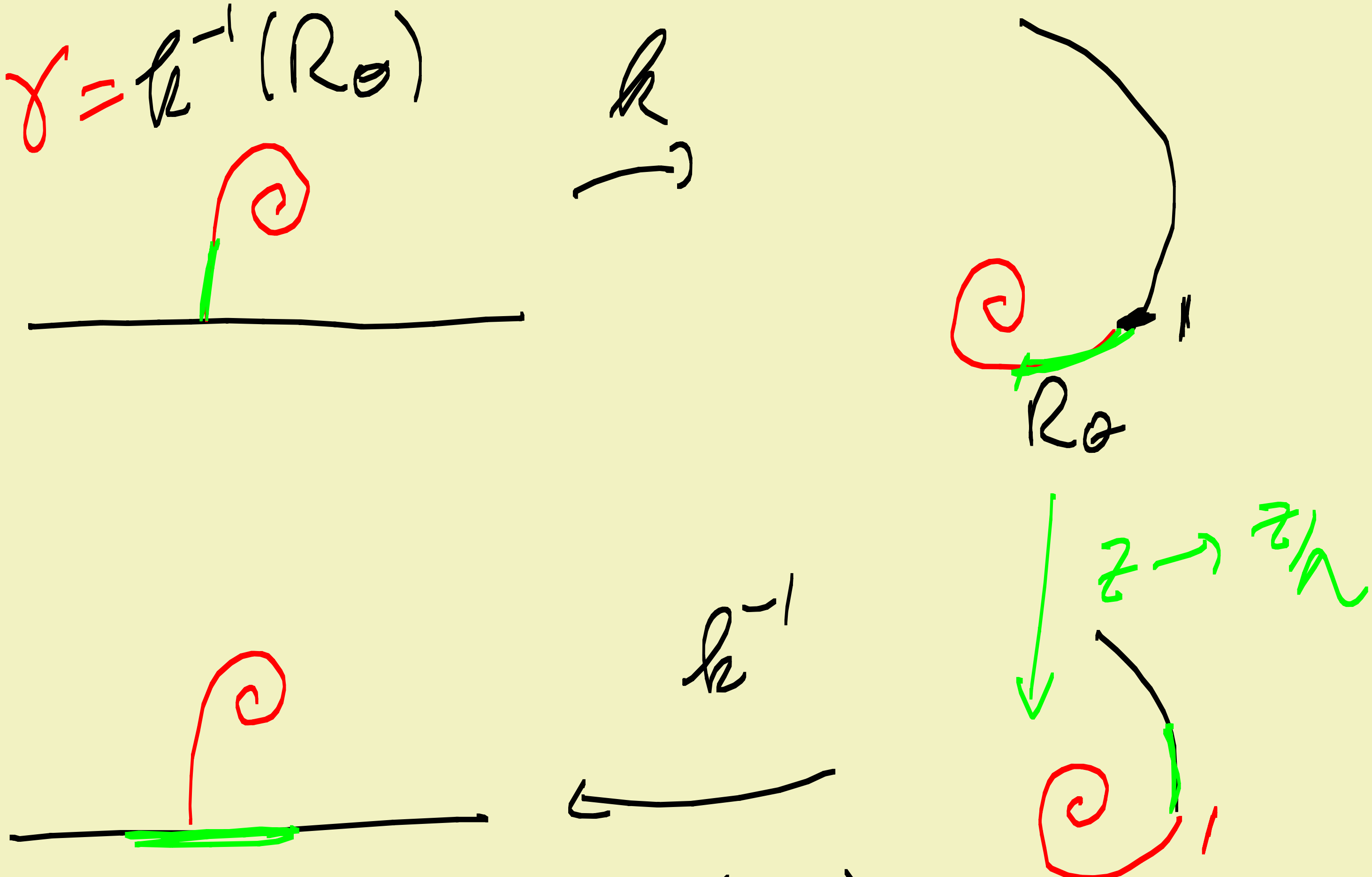
$\gamma = k^{-1}(R_\theta)$





γ = k⁻¹(R_θ)
k
1
R_θ
z → z/λ
k⁻¹
1
G = k⁻¹(1/λ h)

$\gamma = k^{-1}(R_\theta)$



$G = k^{-1}\left(\frac{1}{\lambda} R_\theta\right)$

By prop: $\sigma(\lambda) = K$ so $\lambda(t) = K\sqrt{1-t}$

$$k(z) = C(z-\beta)(z-\bar{\beta})^{e^{2i\theta}} \qquad \beta \in \mathbb{H}$$

$$G = k^{-1}\left(\tfrac{1}{\lambda} k(z)\right) \qquad \lambda = K\sqrt{1-t}$$

PLUG INTO L.D.E. (OR NORMALIZE AT ∞)

$$\beta = 2ie^{i\theta}$$

$$K = -4\sin\theta$$

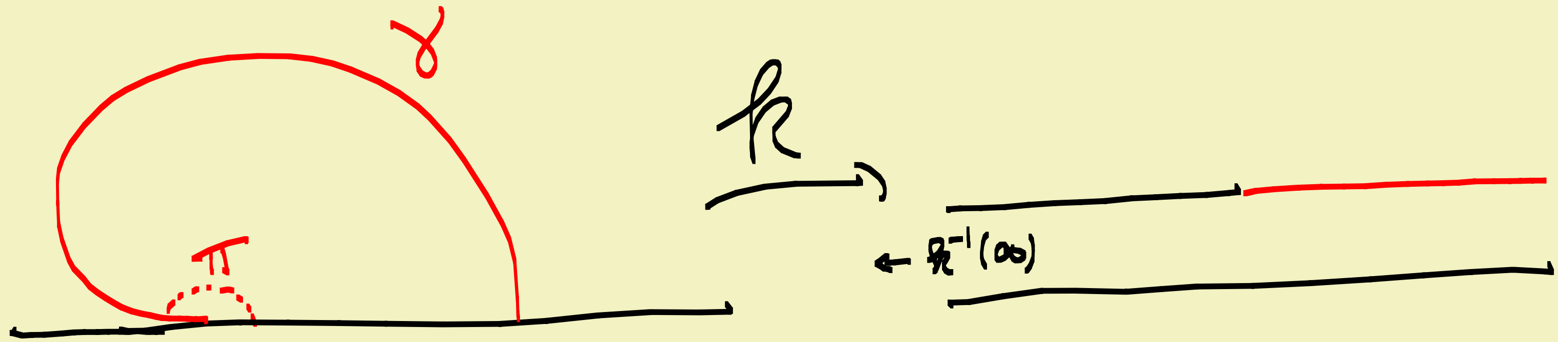
$$0 < |K| < 4$$

CONFORMALITY OF z^{α} (α COMPLEX):

PRETENTENTOONSTELLING
M.C. ESCHER

SEE
SMIT & LENSTRA
NOTICES A.M.S. (50)
2003

LIMITING CASE $K = \pm 4$



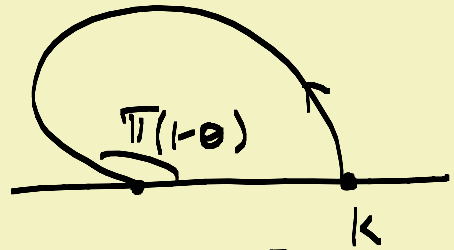
$$z \longrightarrow z - r \qquad (r > 0)$$

$$h(z) = \frac{2}{2-z} - \log(2-z)$$

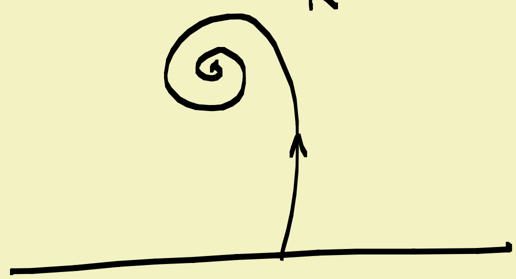
SUMMARY:

K-N-K

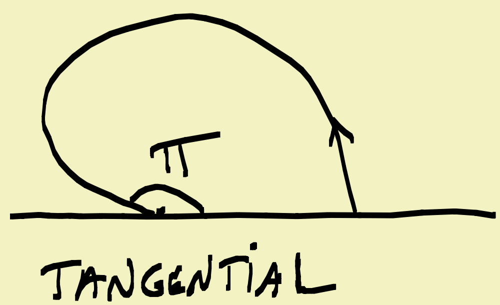
$|K|>4$



$|K|<4$



$|K|=4$

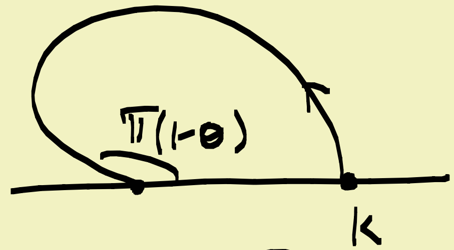


SUMMARY:

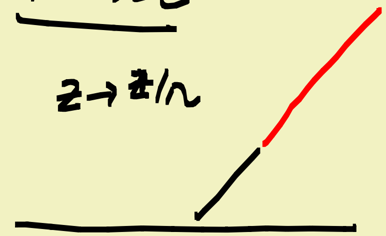
K-N-K

MODEL

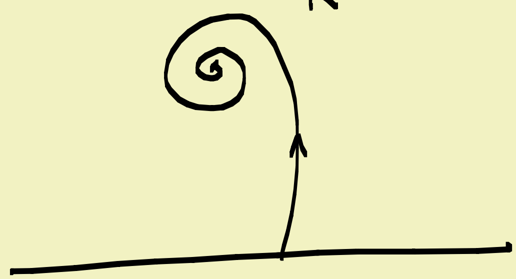
$|K| > 4$



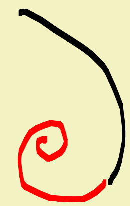
$z \to z/\lambda$



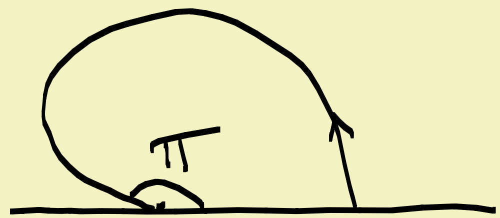
$|K| < 4$



$z \to z/\lambda$

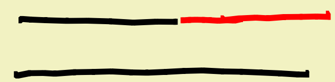


$|K| = 4$

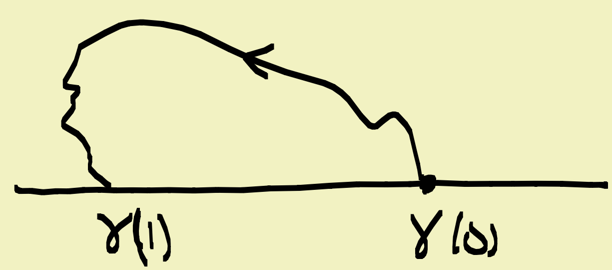


TANGENTIAL

$z \to z-\lambda$



WE'D LIKE CONDITIONS ON λ FORCING γ TO "COLLIDE" WITH $\mathbb{R}$ WHEN $t=1$.



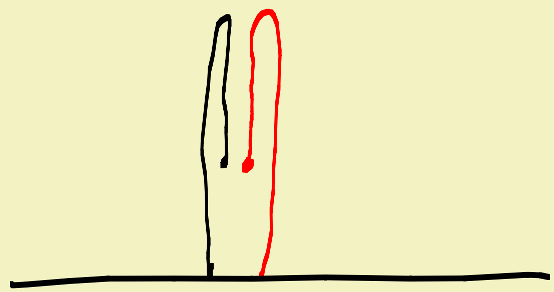
THEOREM (M-ROHDE [2005])

$$|\lambda(t)-\lambda(s)| \le c\,|t-s|^{1/2} \qquad c<c_0$$

$$\Longrightarrow \quad \gamma \text{ is a } \underline{\text{CURVE}} \text{ in } \mathbb{H}$$

J. Lind: $c_0 = 4$.

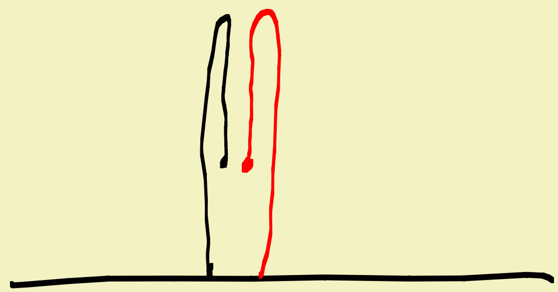
EXAMPLES



$|\gamma_b - \gamma_n|$ small

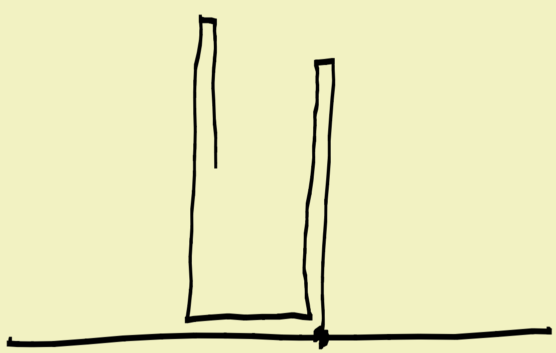
$|\lambda_b - \lambda_n|$ big.

EXAMPLES



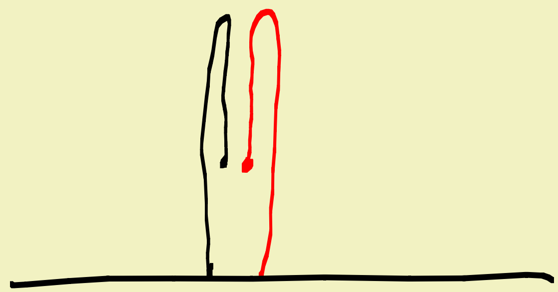
$|\gamma_\ell - \gamma_r|$ small

$|\lambda_\ell - \lambda_r|$ big.



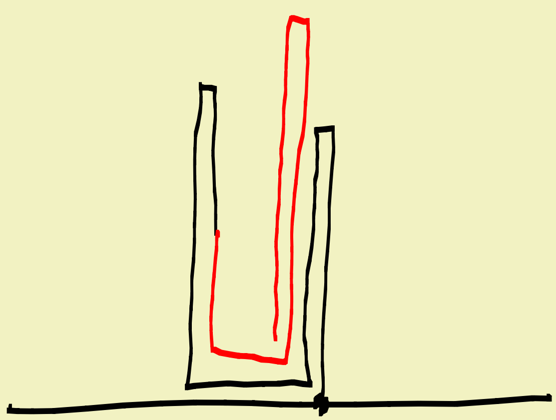
$\lambda_n \to \lambda \quad \not\Rightarrow \quad \gamma^{\lambda_n} \to \gamma^{\lambda}$ (uniformly)

EXAMPLES



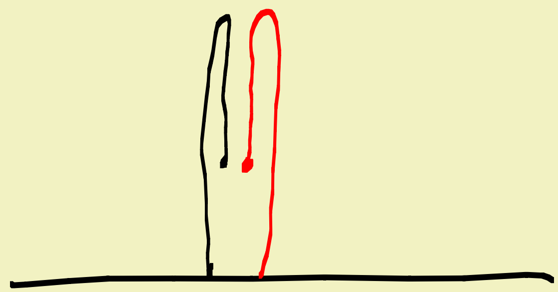
$|\gamma_\ell - \gamma_r|$ small

$|\lambda_\ell - \lambda_r|$ big.



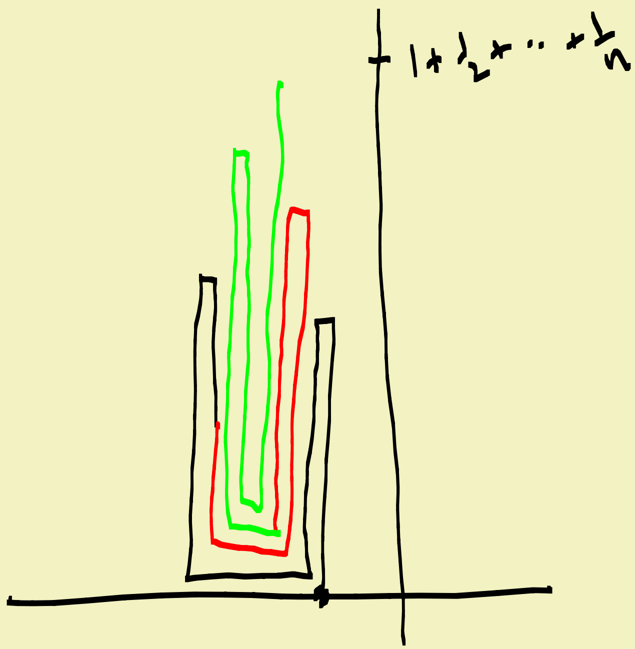
$\lambda_n \to \lambda \not\Rightarrow \gamma^{\lambda_n} \to \gamma^{\lambda}$ (uniformly)

EXAMPLES



$|\gamma_\ell - \gamma_n|$ small

$|\lambda_\ell - \lambda_n|$ big.



rescale to height 1 $\quad \gamma_n$

$\gamma_n \longrightarrow [0, i]$ (Caratheodory)

$$\liminf_{0\le t\le 1} \gamma_n(t) = 0$$

$\lambda_n \to \lambda \not\Rightarrow \gamma^{\lambda_n} \longrightarrow \gamma^{\lambda}$ (uniformly)

THEOREM 1 (LIND-M-ROHDE)

SUPPOSE ① $\lim\limits_{t \to 1} \frac{\lambda(t)}{\sqrt{1-t}} = K > 4$

AND

② $\exists\, \epsilon > 0,\ M < \infty \ni:$

$$|\lambda(t) - \lambda(t')| \leq M \left| \sqrt{1-t} - \sqrt{1-t'} \right|$$

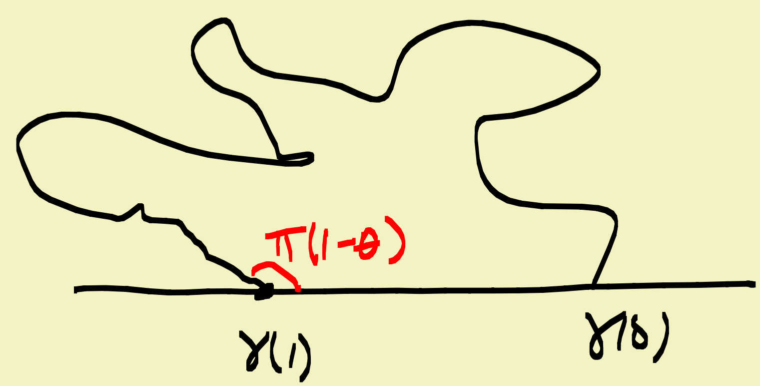
for $1 - \epsilon \leq \frac{1-t'}{1-t} \leq 1$

THEN:

$\gamma : [0,1) \longrightarrow \mathbb{H}$ is A CURVE

$\gamma_T = g_T(\gamma)$ AND $\gamma_T(t) \xrightarrow[t \to 1]{} \mathbb{R}$ FORMING ANGLE $\pi(1-\theta)$ $(u \geq u_0)$

WHERE $K = 2\left(\sqrt{1-\theta} + \frac{1}{\sqrt{1-\theta}}\right)$

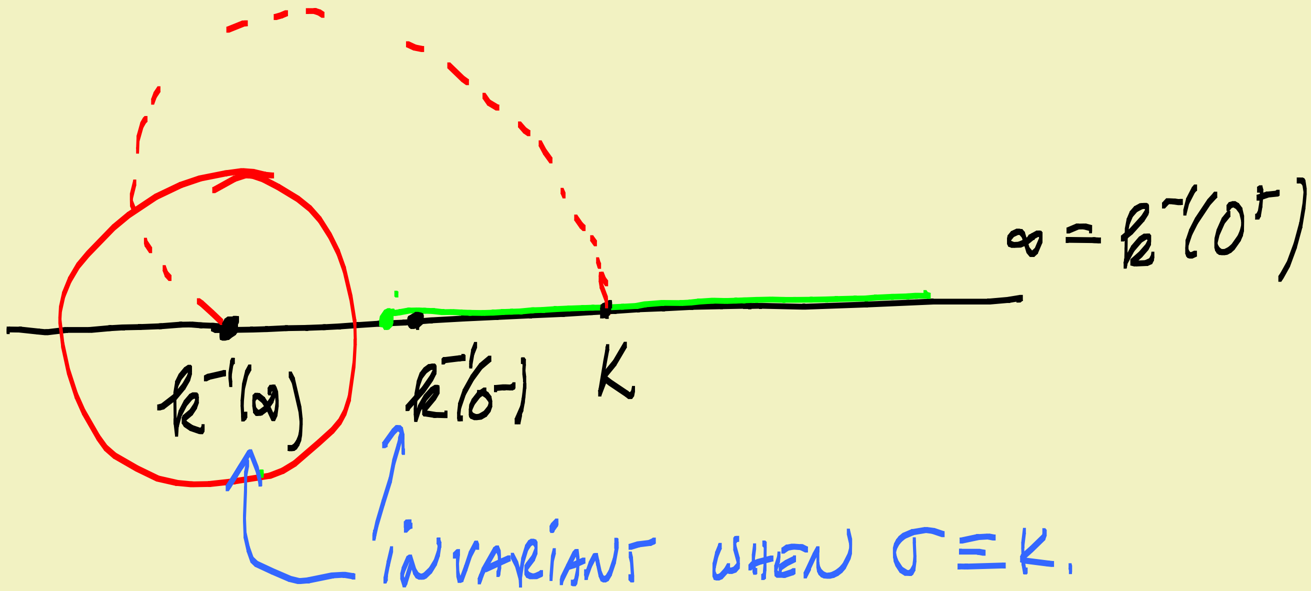


CONDITION 2 $\Longrightarrow$ $\gamma_S(s)$ is a CURVE FOR $0 \le s \le \epsilon$ and <u>all</u> S

(SINCE LIP-$\frac{1}{2}$ NORM < 1)

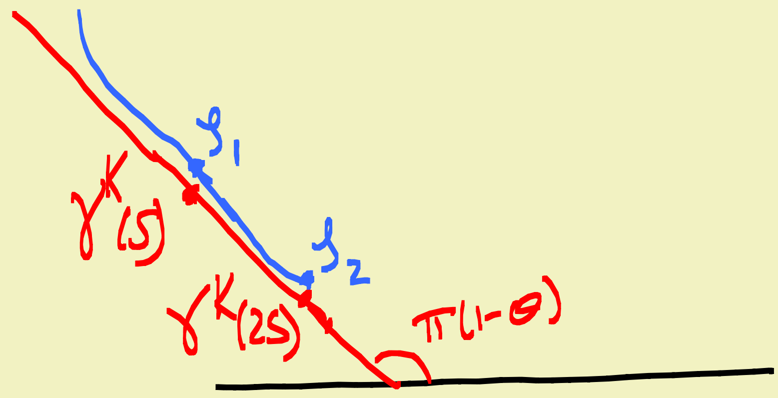
$\therefore$ $\gamma(s(t))$ is a CURVE $0 \le t < 1$

1 $\Longleftrightarrow$ $\sigma_s \to K$ UNIFORMLY



$$G_\Delta\left([\gamma(0), \gamma(s)]\right) > h^{-1}(\sigma^-) - \delta \qquad \text{if } |\sigma - K| \text{ SMALL}$$

$$G_\Delta^{-1} = F_\Delta \text{ univalent on } B\left(k^{-1}(\infty), \delta_0\right)$$



$\gamma \longleftrightarrow \sigma$

$\gamma^K \longleftrightarrow \sigma \equiv K$

$G_s(\gamma(s+v)) \sim \gamma^K(v)$

$0 \le v \le v_0$

Lemma

$n \geqslant n_0$

$\zeta_n = \gamma(nS)$ $\quad$ u large

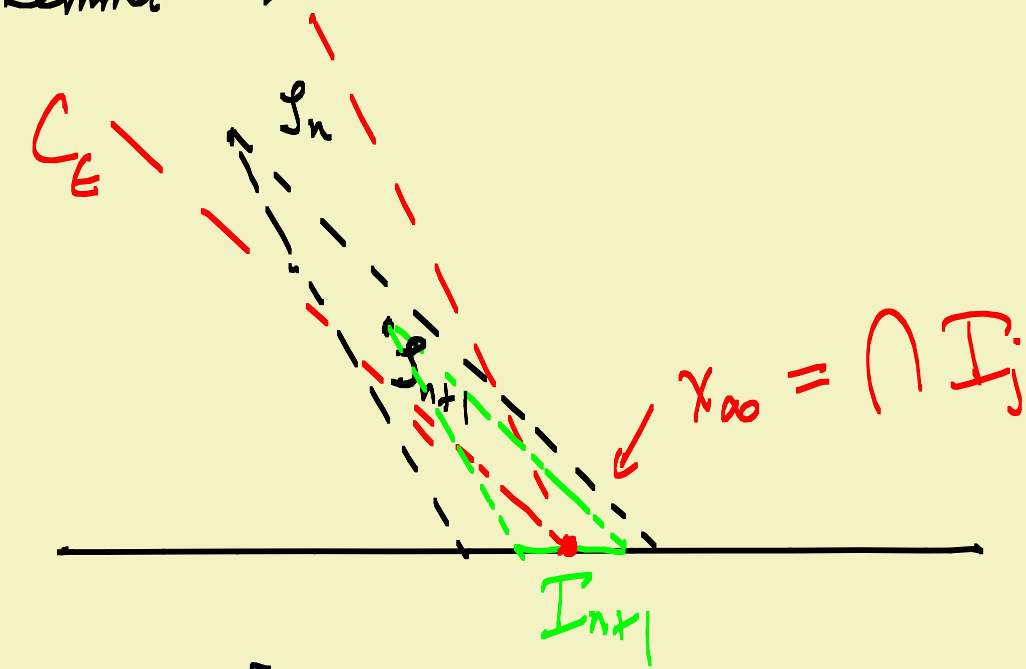
① $\left|\arg(\zeta_n - \zeta_{n+1}) - \pi(1-\theta)\right| < \epsilon$

② $\operatorname{Im} \zeta_{n+1} < \frac{1}{2} \operatorname{Im} \zeta_n$

③ $\rho_{\mathbb{H}}\left(\gamma(s), [\zeta_n, \zeta_{n+1}]\right) < \epsilon$

$nS \le s \le (n+1)S$

Lemma $\Rightarrow$ Theorem



$$[g_n, g_{n+1}] \subset x_\infty + C_\epsilon$$

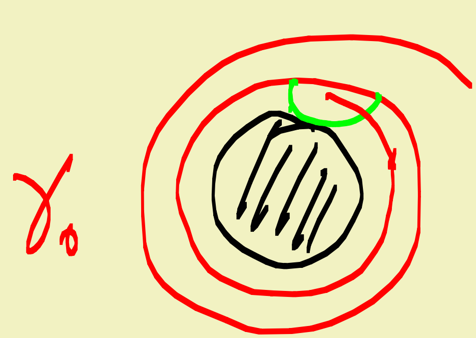
IF $\dfrac{\lambda(t)}{\sqrt{1-t}} \longrightarrow K < 4$

(and 2 HOLDS)

THEN $\gamma[0,1]$ is A CURVE in $\mathbb{H}$

BY M-R RESULT.

WHAT ABOUT $K = 4$?

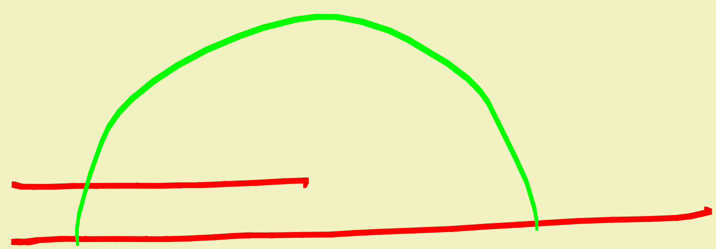


$$r = \frac{1}{\theta} + 1 \qquad z = re^{i\theta}$$

$$r(\theta) - 1 = \frac{1}{\theta}$$

$$r(\theta) - r(\theta + 2\pi) = \frac{2\pi}{\theta^2}$$

NEAR TIP LOOKS LIKE A ROTATION OF:



(CF. K=4 MODEL)

K compact connected

$\varphi : \mathbb{D}^c \to K^c$

$\varphi(\gamma_0)$ has similar property

THEOREM 2 (LIND–M–ROHDE)

L COMPACT, CONNECTED $\subset \mathbb{H}$

THEN $\exists$ SPIRAL $S \subset \mathbb{H} \cup \{0\} \setminus L$

WITH $S \to L$

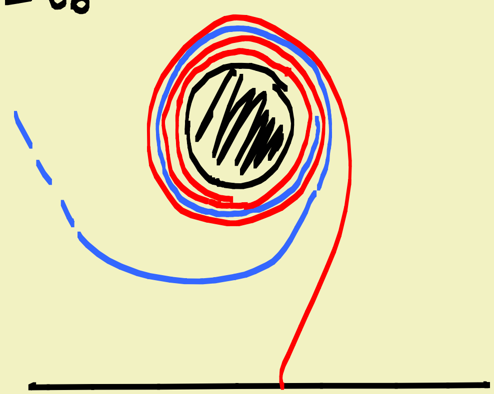
$$\frac{\lambda(t)}{\sqrt{t_0 - t}} \longrightarrow 4$$

$$|\lambda(t) - \lambda(t')| \leq M \left| \sqrt{t_0 - t} - \sqrt{t_0 - t'} \right|$$

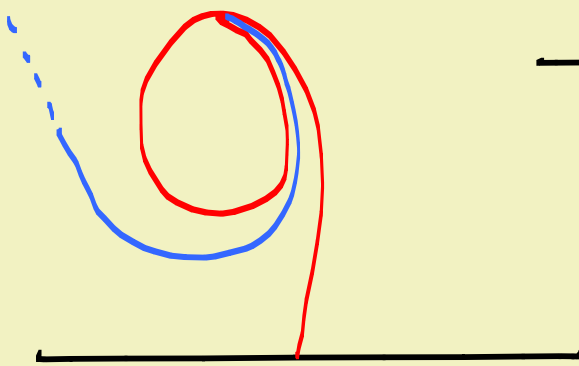
$$\text{for } 1-\epsilon \leq \frac{t_0 - t'}{t_0 - t} \leq 1$$

COR. THE TRACE γ FOR λ IS NOT CONT. ON $[0,1]$ BUT THE TRACE FOR $c\lambda$ ($c \neq \pm 1$) IS CONTINUOUS ON $[0,1]$.

λ as above $t \le 1 = t_0$
$\lambda \approx 0$ for $t > 1$

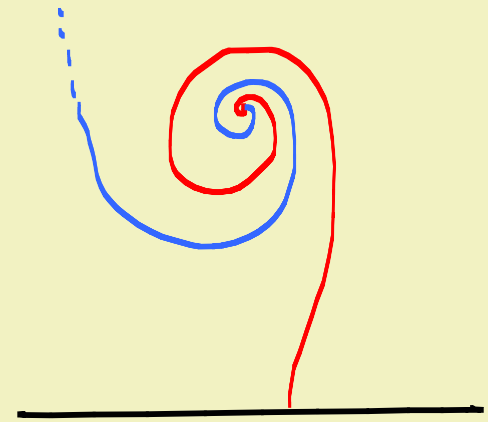


$\gamma \leftrightarrow \lambda$



$\gamma \leftrightarrow c\lambda$

$c > 1$



$\gamma \leftrightarrow c\lambda$

$c < 1$

Idea:

$$\gamma_2 \subset \gamma_1 + B(0,\epsilon)$$

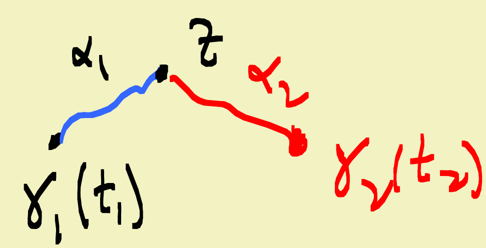
$$\operatorname{diam} \gamma_j \le M$$

$$\gamma_1 \subset \gamma_2 + B(0,\epsilon)$$

$$\exists\ z : \operatorname{dist}(z, \gamma_j) \geq \epsilon$$

$\exists$ curves α_j

$\alpha_j \subset \mathbb{H} \setminus \gamma_j$



$\ni$: $\operatorname{diam}(\alpha_j) \le C\epsilon$

then $|\lambda_1(t_1) - \lambda_2(t_2)| \leq C(\epsilon, M)$

THANKS!