

Operator semigroups arising from semigroups of analytic functions

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1. Introduction

$\mathbf{D} = \{z : |z| < 1\}$ unit disc in \mathbf{C} .

$\mathbf{H}(\mathbf{D}) = \{f : \mathbf{D} \rightarrow \mathbf{C}, f \text{ analytic}\},$

$\mathbf{B}(\mathbf{D}) = \{\phi \in \mathbf{H}(\mathbf{D}) : |\phi(z)| < 1\},$ semigroup under composition.

Def. $\{\phi_t\}_{t \geq 0} \subset \mathbf{B}(\mathbf{D})$ is a one-parameter semigroup if

$$t \mapsto \phi_t : \mathbf{R}^+ \rightarrow \mathbf{B}(\mathbf{D}),$$

is a continuous homomorphism.

Thus:

- $\phi_t(\mathbf{D}) \subset \mathbf{D}, \phi_0(z) \equiv z,$
- $\phi_{t+s} = \phi_t \circ \phi_s, \text{ all } t, s \geq 0,$
- $\phi_t \rightarrow \phi_0 \equiv z, \text{ as } t \rightarrow 0, \text{ uniformly on compact subsets of } \mathbf{D}.$

Examples.

- $\phi_t(z) = e^{it}z, \quad \phi_t(z) = e^{-t}z,$
- $\phi_t(z) = e^{-t}z + 1 - e^{-t},$
- $\phi_t(z) = \frac{e^{-t}z}{(e^{-t}-1)z+1},$
- $\phi_t(z) = 1 - (1-z)e^{-t},$
- $\phi_t(z) = \frac{(1+e^t)z-1+e^t}{(-1+e^t)z+1+e^t},$
- $\phi_t(z) = h^{-1}(e^{-t}h(z)), \quad h : \mathbf{D} \rightarrow \mathbf{C},$
- $\phi_t(z) = k^{-1}(k(z) + t), \quad k : \mathbf{D} \rightarrow \mathbf{C},$

where h, k are appropriate univalent maps on \mathbf{D} .

Given such a semigroup define

$$T_t(f) = f \circ \phi_t,$$

for $f \in H(\mathbf{D})$. The family $\{T_t\}_{t \geq 0}$ satisfies

- $T_0 \equiv I$, the identity operator,
- $T_t \circ T_s = T_{t+s}, \quad t, s \geq 0,$

i.e. it forms a semigroup of operators on $H(\mathbf{D})$.

If $X \subset H(\mathbf{D})$, a linear (Banach) space, such that the composition operators $T_t : X \rightarrow X$ are bounded, then we have an operator semigroup $\{T_t\}_{t \geq 0}$ on X .

We review the main results about $\{T_t\}$, concentrating on Operator Theory (rather than Function Theory). We confine ourselves in 1-dimensional case even though semigroups $\{\phi_t\}$ have been defined in several (even infinite) dimensions. Questions to be discussed are:

- **Strong continuity.** When is $\{T_t\}$ strongly continuous on X ?
- **Spectral theory.** Infinitesimal generator, Resolvent operator, compactness, spectra.
- **Weighted composition semigroups and applications.**

Whenever possible we try to find the relation between function theoretic properties of $\{\phi_t\}$ and operator theoretic properties of $\{T_t\}$.

2. Notation and background

The semigroups $\{T_t\}$ were first studied by E. Berkson - H. Porta (1978), on Hardy spaces, and later by other authors on Hardy and other spaces. We recall some basic properties of $\{\phi_t\}$:

- each ϕ_t is univalent.
- Infinitesimal generator

$$G(z) = \lim_{t \rightarrow 0^+} \frac{\phi_t(z) - z}{t},$$

uniformly on compact subsets of \mathbf{D} .

- $G(z)$ satisfies

$$G(\phi_t(z)) = \frac{\partial \phi_t(z)}{\partial t} = G(z) \frac{\partial \phi_t(z)}{\partial z},$$

for each $z \in \mathbf{D}$, $t \geq 0$.

- G has a unique representation

$$G(z) = (\bar{b}z - 1)(z - b)F(z),$$

where $|b| \leq 1$, $F \in H(\mathbf{D})$ with $\operatorname{Re} F(z) \geq 0$.

b is the Denjoy-Wolff point of $\{\phi_t\}$.

Other characterizations for the infinitesimal generator that can be found for example in the book of D. Shoikhet.

The cases $|b| < 1$ and $|b| = 1$ turn out to be distinctly different. For simplicity, and often w.l.o.g., we may assume:

$b = 0$, for semigroups with $|b| < 1$,

$b = 1$, for semigroups with $|b| = 1$.

In fact we can achieve this by composing ϕ_t with automorphisms of \mathbf{D} .

Further to each $\{\phi_t\}$ there corresponds a unique associated univalent function $h : \mathbf{D} \rightarrow \mathbf{C}$ obtained as:

Case $b = 0$. Let h be the solution of

$$\frac{h'(z)}{h(z)} = \frac{G'(0)}{G(z)}, \quad h(0) = 0.$$

Because $\operatorname{Re} F(z) \geq 0$, it can be shown that h is a univalent *spirallike* function and Schröder's functional equation is satisfied

$$h(\phi_t(z)) = e^{ct} h(z), \quad z \in \mathbf{D}, \quad t \geq 0,$$

where $c = G'(0)$.

Case $b = 1$. Let h be defined by

$$h'(z) = \frac{G(0)}{G(z)}, \quad h(0) = 0.$$

This h is a *close-to-convex* univalent function, having the geometric property that it is "starlike at ∞ in a certain direction", and Abel's functional equation holds

$$h(\phi_t(z)) = h(z) + ct, \quad z \in \mathbf{D}, \quad t \geq 0.$$

with $c = G(0)$.

Thus, in principle, all information about $\{\phi_t\}$ is contained in any of the following objects:

- the generator $G(z)$
- the pair (b, F)
- the pair (b, h)
- the pair (b, μ) where μ is the Borel measure in the Herglotz representation of $1/F$,

$$\frac{1}{F(z)} = \int_{\partial\mathbf{D}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta) + i \operatorname{Im} \frac{1}{F(0)}.$$

1. Strong continuity. Suppose $(X, \| \cdot \|)$ is a Banach space as above. Strong continuity of $\{T_t\}$ requires

$$\lim_{t \rightarrow 0} \|T_t(f) - f\|_X = 0, \text{ for each } f \in X,$$

i.e. need to pass from pointwise convergence

$$\lim_{t \rightarrow 0} f \circ \phi_t(z) = f(z), \quad z \in \mathbf{D},$$

to convergence in $\| \cdot \|_X$.

A general argument:

Assume X contains the polynomials as dense set, then for a polynomial P we have,

$$\begin{aligned} \|f \circ \phi_t - f\| &\leq \\ &\leq \|f \circ \phi_t - P \circ \phi_t\| + \|P \circ \phi_t - P\| + \|P - f\| \\ &\leq (\|T_t\| + 1)\|P - f\| + \|P \circ \phi_t - P\|. \end{aligned}$$

Assume further $\sup_{t \in (0, \delta)} \|T_t\| < \infty$ for some $\delta > 0$, then we only need

$$\lim_{t \rightarrow 0} \|P \circ \phi_t - P\| = 0$$

for each P , and this will follow if we can show

$$\lim_{t \rightarrow 0} \|\phi_t(z) - z\| = 0.$$

The latter condition holds in many classical spaces as a result of the dominated convergence or a similar theorem.

Using this and other similar arguments, one can show strong continuity on many classical spaces. We collect the known cases:

- Every $\{\phi_t\}$ induces strongly continuous $\{T_t\}$ on:

- i) the Hardy spaces H^p , $1 \leq p < \infty$,

- ii) the Bergman spaces A^p , $1 \leq p < \infty$,

- iii) the Dirichlet space \mathcal{D} ,

- iv) $VMOA$ and \mathcal{B}_0 (vanishing mean oscillation and "little Bloch" spaces).

- No nontrivial $\{\phi_t\}$ induces a strongly continuous operator semigroup on H^∞ .

- On the disc algebra $A(\mathbf{D})$ both kinds of behavior can be found. M. Contreras and S. Diaz-Madriral (2005), using the theory of prime ends, gave conditions equivalent to strong continuity of $\{T_t\}$. One such is: $\partial_\infty h(\mathbf{D})$ is locally connected, where $h =$ the associated univalent function to $\{\phi_t\}$.

- On $BMOA$ the situation is more complicated. Given $\{\phi_t\}$ we would need

$$\lim_{t \rightarrow 0} \|f \circ \phi_t - f\|_* = 0, \quad \text{for each } f \in BMOA,$$

but an old theorem of D. Sarason says:

Thm. For $f \in BMOA$, following are equivalent:

(a) $f \in VMOA$.

(b) $\lim_{t \rightarrow 0} \|f(e^{it}z) - f(z)\|_* = 0$.

(c) $\lim_{t \rightarrow 0} \|f(e^{-t}z) - f(z)\|_* = 0$.

Thus for $\phi_t(z) = e^{it}z$ or $\phi_t(z) = e^{-t}z$, not a single $f \in BMOA \setminus VMOA$ obeys strong continuity.

Maybe this is the case for an arbitrary $\{\phi_t\}$?

Ex. Let $\phi_t(z) = e^{-t}z + 1 - e^{-t}$ and

$$f_0(z) = \log \frac{1}{1-z} \in BMOA.$$

Then

$$\lim_{t \rightarrow 0} \|f_0 \circ \phi_t - f_0\|_* = \dots = \lim_{t \rightarrow 0} (t) = 0,$$

but $f_0 \notin VMOA$. Thus $\{\phi_t\}$ induces a strongly continuous $\{T_t\}$ on the space

$$V = \text{span}\{VMOA, f_0\}.$$

Ex. Let $h : \mathbf{D} \rightarrow \mathbf{C}$ be starlike univalent such that $h(0) = 0$ and $h \in BMOA \setminus VMOA$, and let

$$\phi_t(z) = h^{-1}(e^{-t}h(z)).$$

Then

$$\lim_{t \rightarrow 0} \|h \circ \phi_t - h\|_* = \lim_{t \rightarrow 0} |e^{-t} - 1| \|h\|_* = 0,$$

so $\{T_t\}$ is strongly continuous on $\text{span}\{VMOA, h\}$.

We are led to the conclusion: given $\{\phi_t\}$ there is a "largest" subspace of $BMOA$, on which $\{T_t\}$ is strongly continuous. Denote this subspace by $[\phi_t, BMOA]$.

Clearly

$$VMOA \subseteq [\phi_t, BMOA]$$

for every $\{\phi_t\}$, and there are $\{\phi_t\}$ such that

$$VMOA \subsetneq [\phi_t, BMOA].$$

By Sarason's theorem, if $\phi_t(z) = e^{it}z$ or $\phi_t(z) = e^{-t}z$, then

$$[\phi_t, BMOA] = VMOA.$$

The problem arises to identify those $\{\phi_t\}$ of extremal behavior, i.e.

$$VMOA = [\phi_t, BMOA], \text{ or, } BMOA = [\phi_t, BMOA].$$

In a joint paper with O. Blasco, M. Contreras, S. Diaz-Madrigal and J. Martinez (2008) we proved

Thm. Let $\{\phi_t\}$ with generator G . Assume there is an $0 < \alpha < 1$, such that

$$\frac{(1 - |z|)^\alpha}{G(z)} = O(1), \quad |z| \rightarrow 1.$$

Then $VMOA = [\phi_t, BMOA]$.

Note that, for every semigroup with DW point $b \in \mathbf{D}$, we have

$$\frac{1 - |z|}{G(z)} = O(1), \quad |z| \rightarrow 1.$$

due to the growth inequalities for $F(z)$.

There is partial converse:

Thm. Suppose $\{\phi_t\}$ with generator G has the DW point $b \in \mathbf{D}$, and is such that $VMOA = [\phi_t, BMOA]$. Then

$$\lim_{|z| \rightarrow 1} \frac{1 - |z|}{G(z)} = 0.$$

One can further show that if $VMOA = [\phi_t, BMOA]$ then the Herglotz measure representing $1/F$ has no point masses on $\partial\mathbf{D}$.

2. Spectral theory. If $\{T_t\}$ is strongly continuous on X , its infinitesimal generator Γ is the (unbounded) operator

$$\Gamma(x) = \lim_{t \rightarrow 0} \frac{T_t(x) - x}{t},$$

the limit taken in $\|\cdot\|_X$. This limit exists for x in a linear manifold $D(\Gamma) \subseteq X$ (the domain of Γ). The equality $D(\Gamma) = X$ happens exactly when the semigroup is uniformly continuous i.e.

$$\lim_{t \rightarrow 0} \|T_t - I\|_X = 0,$$

and then $\Gamma : X \rightarrow X$ is bounded.

In our case, given $\{\phi_t\}$, the generator of $\{T_t\}$ is

$$\Gamma(f) = \lim_{t \rightarrow 0} \frac{T_t(f) - f}{t} = \lim_{t \rightarrow 0} \frac{f \circ \phi_t - f}{t},$$

and because convergence in norm, in the usual classical spaces, implies pointwise convergence, we find for each $f \in D(\Gamma)$ and $z \in \mathbf{D}$,

$$\begin{aligned} \Gamma(f)(z) &= \left. \frac{\partial f(\phi_t(z))}{\partial t} \right|_{t=0} \\ &= f'(z) \left. \frac{\partial \phi_t(z)}{\partial t} \right|_{t=0} \\ &= f'(z)G(z) \end{aligned}$$

where G is the generator of $\{\phi_t\}$. One can then complete the technical details to show indeed

$$\Gamma(f) = Gf',$$

with $D(\Gamma) = \{f \in X : Gf' \in X\}$, and that Γ is bounded only for the trivial semigroup: $\phi_t(z) = z$, $t \geq 0$, which corresponds to $G \equiv 0$.

For the point spectrum $\sigma_p(\Gamma)$ solve

$$Gf' = \lambda f$$

to find

$$\begin{aligned} \sigma_p(\Gamma) &= \{-ck : h(z)^k \in X\}, & (\text{DW point} = 0) \\ &= \{c\lambda : e^{\lambda h(z)} \in X\}, & (\text{DW point} = 1) \end{aligned}$$

where h is the univalent function associated to $\{\phi_t\}$.

Note $\sigma_p(\Gamma)$ in the second case can be large. For example suppose $X = H^2$ and

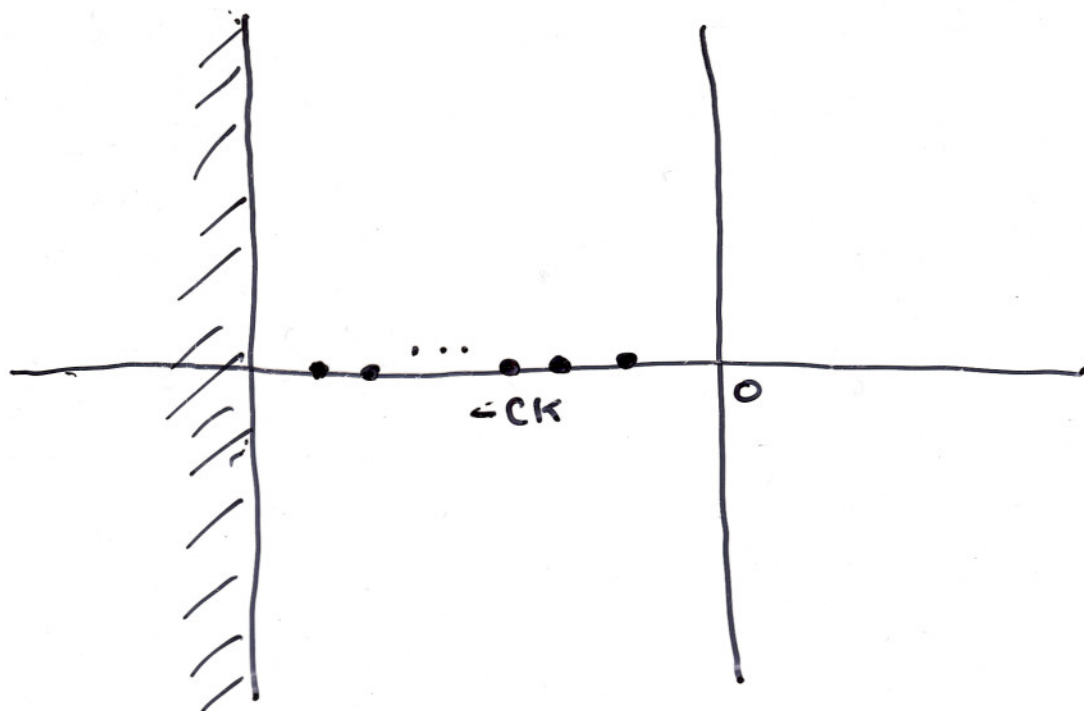
$$h(\mathbf{D}) \subset \text{strip}$$

then $h \in BMOA$ and this implies $e^h \in H^p$ for some $p > 0$, so $\sigma_p(\Gamma)$ contains a disc around 0.

The full spectrum $\sigma(\Gamma)$ can contain large part of the plane even when the DW point is 0. Indeed its typical shape is as follows

Typical shape of $\sigma(\Gamma)$
(DWR point = 0)

(14)



$\sigma(\Gamma) = \{\text{isolated eigenvalues}\}$

$\cup \{\text{part of half-plane } \operatorname{Re}(z) \leq s\}$

$s = s(h)$ depends on h .

Examples, DW point = 0 (on $X = H^p$)

1. For $\phi_t(z) = e^{-ct}$,

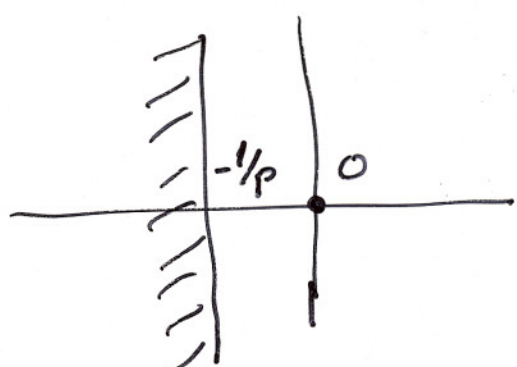
$$\sigma_p(\Gamma) = \sigma(\Gamma) = \{-kc : k = 0, 1, 2, \dots\}.$$

2. For $\phi_t(z) = 1 - (1 - z)e^{-t}$,

$$\sigma_p(\Gamma) = \sigma(\Gamma) = \{-k : k = 0, 1, 2, \dots\}$$

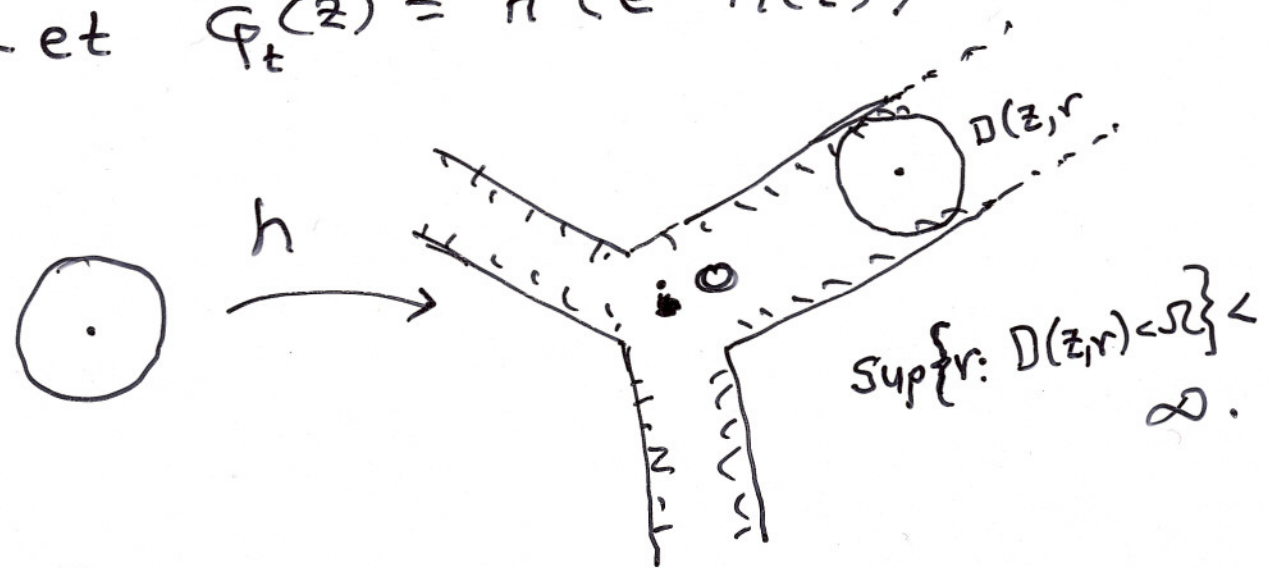
3. Let $\phi_t(z) = \frac{e^{-t}z}{(e^{-t}-1)z+1}$.

Here $h(z) = \frac{z}{1-z}$ and we find $\sigma_p(\Gamma) = \{0\}$. A technical calculation gives



$$\sigma(\Gamma) = \{0\} \cup \left\{ \text{Re}(z) \leq -\frac{1}{p} \right\}$$

4. Let $\varphi_t(z) = h^{-1}(e^{-t}h(z))$



Then

$$\sigma(\Gamma) = \{0, -1, -2, \dots\}.$$

Examples, DW point = 1

1. Let $\phi_t(z) = e^{-t}z + 1 - e^{-t}$. This has

$$h(z) = \log \frac{1}{1-z}.$$

Now $e^{\lambda h(z)} = (1-z)^{-\lambda} \in H^p \Leftrightarrow \operatorname{Re}(\lambda) < 1/p$,
hence

$$\sigma_p(\Gamma) = \{z : \operatorname{Re}(z) < 1/p\}.$$

Using norm estimates for $\|T_t\|$ and the fact that $\sigma(\Gamma)$ is a closed set we obtain

$$\sigma(\Gamma) = \{z : \operatorname{Re}(z) \leq 1/p\}.$$

2. For the group of hyperbolic automorphisms

$$\phi_t(z) = \frac{(1+e^t)z - 1 + e^t}{(-1+e^t)z + 1 + e^t}.$$

with associated univalent function $h(z) = \log \frac{1+z}{1-z}$
we have

$$e^{\lambda h(z)} = \left(\frac{1+z}{1-z}\right)^\lambda \in H^p \Leftrightarrow -1/p < \operatorname{Re}(\lambda) < 1/p.$$

It follows that

$$\sigma_p(\Gamma) = \{z : -1/p < \operatorname{Re}(z) < 1/p\}.$$

As in the previous example we conclude

$$\sigma(\Gamma) = \{z : -1/p \leq \operatorname{Re}(z) \leq 1/p\}.$$

Compactness of resolvent. (when the DW point is 0)

The resolvent operator is

$$R(\lambda, \Gamma) = (\lambda - \Gamma)^{-1}, \quad \lambda \in \rho(\Gamma).$$

Using the resolvent equations, we may calculate the resolvent at a convenient point. We chose $\lambda = G'(0) \in \rho(\Gamma)$. Then $R(G'(0), \Gamma)$ is a constant multiple of

$$R_h(f)(z) = \frac{1}{h(z)} \int_0^z f(\zeta) h'(\zeta) d\zeta.$$

It can be shown that this is a compact operator if and only if the operator

$$Q_h(f)(z) = \frac{1}{z} \int_0^z f(\zeta) \frac{\zeta h'(\zeta)}{h(\zeta)} d\zeta,$$

is compact (in fact they belong, or do not belong, simultaneously to every operator ideal).

Now from the identity

$$zh'(z)/h(z) = 1 + z(\log \frac{h(z)}{z})'$$

we have

$$\begin{aligned} Q_h(f)(z) &= \frac{1}{z} \int_0^z f(\zeta) d\zeta + \frac{1}{z} \int_0^z f(\zeta) \zeta \left(\log \frac{h(\zeta)}{\zeta} \right)' d\zeta \\ &= J(f)(z) + L_h(zf)(z), \end{aligned}$$

where J =integration and L_h is

$$L_h(f)(z) = \frac{1}{z} \int_0^z f(\zeta) \left(\log \frac{h(\zeta)}{\zeta} \right)' d\zeta.$$

Recall that if h is univalent with $h(0) = 0$ then

$$\log \frac{h(z)}{z} \in BMOA,$$

and then a lemma from a 1977 paper of Pommerenke says

$$L_h \text{ compact on } H^2 \Leftrightarrow \log \frac{h(z)}{z} \in VMOA.$$

Since J is compact on H^2 we conclude that, on H^2 ,

$$R(\lambda, \Gamma) \text{ compact} \Leftrightarrow \log \frac{h(z)}{z} \in VMOA.$$

From the compactness behavior of single composition operators, together with a certain theorem relating the compactness of the resolvent to the compactness of the individual members of a semigroup, one is led to suspect that the same

kind of characterization should hold on other $H^p, p \neq 2$. But to prove this one needs to consider the operators

$$T_g(f)(z) = \int_0^z f(\zeta)g'(\zeta) d\zeta$$

and study their boundedness or compactness properties on Hardy spaces. Fortunately what we need is true:

Thm. (A. Aleman, A.S.) Let $1 \leq p < \infty$, then

(i) T_g bounded on $H^p \Leftrightarrow g \in BMOA$

(ii) T_g compact on $H^p \Leftrightarrow g \in VMOA$.

Thus we have

Thm. Suppose $1 \leq p < \infty$ then the following are equivalent:

(i) $R(\lambda, \Gamma)$ compact on H^2

(ii) $R(\lambda, \Gamma)$ compact on H^p

(iii) $\log \frac{h(z)}{z} \in VMOA$.

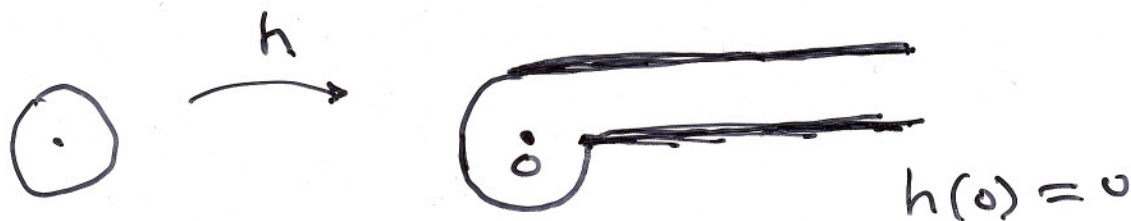
It can be shown further that each of these conditions is equivalent to

(iv) The Herglotz measure of $\frac{1}{F}$ has no point masses.

(v) $h \in \cap_{p < \infty} H^p$.

Similar things about $R(\lambda, \Gamma)$ can be proved on other spaces (Bergman spaces e.t.c.)

Remark. Can construct composition semigroup such that no T_t is compact but $R(\lambda, \Gamma)$ is compact. Indeed take the Riemann map h which maps \mathbf{D} onto the starlike region



Let $\phi_t(z) = h^{-1}(e^{-t}h(z))$.

Then $\phi_t(\partial\mathbf{D}) \cap \partial\mathbf{D}$ is an arc (of positive linear measure). This implies that none of T_t is compact.

But this h belongs to $\cap_{p < \infty} H^p$ so that $R(\lambda, \Gamma)$ is compact on H^p .

3. Weighted composition semigroups

Let X be as above and $\{\phi_t\}$ a semigroup. Suppose $w : \mathbf{D} \rightarrow \mathbf{C}$ is analytic such that

$$S_t(f)(z) = \frac{w(\phi_t(z))}{w(z)} f(\phi_t(z)), \quad f \in X,$$

are bounded operators on X . Then $\{S_t : t \geq 0\}$ is an operator semigroup ($w \equiv 1$ corresponds to the unweighted case).

There are many possible choices for w . If the DW point is 0 then $w(z) = z$ is possible and in fact $w(z) = z^r$, (r real) is a possible choice,

$$S_t(f)(z) = \left(\frac{\phi_t(z)}{z} \right)^r f(\phi_t(z))$$

are bounded operators on H^p

More generally let $\{\phi_t\}$ be given. A cocycle for $\{\phi_t\}$ is a family $\{m_t : t \geq 0\}$ of analytic functions $m_t : \mathbf{D} \rightarrow \mathbf{C}$ such that

- $m_0 \equiv 1$,
- $m_{t+s}(z) = m_t(z)m_s(\phi_t(z))$, $z \in \mathbf{D}$, $t, s \geq 0$,
- $t \rightarrow m_t(z)$ is continuous for each $z \in \mathbf{D}$.

For a given $\{\phi_t\}$ and appropriate w , the family

$$m_t(z) = \frac{w(\phi_t(z))}{w(z)},$$

is a cocycle. A coboundary for $\{\phi_t\}$ is a cocycle that can be written in this form for some w . Not all cocycles are coboundaries.

A large class of cocycles for $\{\phi_t\}$ is constructed as follows. Take g analytic on \mathbf{D} and let

$$m_t(z) = \exp \left(\int_0^t g(\phi_s(z)) ds \right).$$

In particular if G is the generator of $\{\phi_t\}$ and w analytic such that $w'(z)G(z)/w(z)$ is also analytic (this allows w to have zeros only at the DW point), then the choice $g = w'G/w$ is possible and

$$\begin{aligned} m_t &= \exp \left(\int_0^t \frac{w' \circ \phi_s}{w \circ \phi_s} G \circ \phi_s \right) \\ &= \exp \left(\int_0^t \frac{\frac{\partial(w \circ \phi_s)}{\partial s}}{w \circ \phi_s} ds \right) \\ &= \frac{w \circ \phi_t}{w}, \end{aligned}$$

so the cocycle obtained for this g is a coboundary.

If $\{m_t\}$ is a cocycle for $\{\phi_t\}$ then

$$U_t(f)(z) = m_t(z)f(\phi_t(z)), \quad f \in X,$$

defines formally a semigroup $\{U_t\}$ of operators on X . If U_t are bounded then they form a semigroup on X and similar questions about strong continuity e.t.c. arise.

For coboundaries $m_t = w \circ \phi_t / w$ the following conditions imply strong continuity on H^p , $1 \leq p < \infty$,

- $\limsup_{t \rightarrow 0} \left\| \frac{w \circ \phi_t}{w} \right\|_\infty \leq 1.$

- $w \in H^q$ for some $q > 0$, and

$$\limsup_{t \rightarrow 0} \left\| \frac{w \circ \phi_t}{w} \right\|_\infty < \infty.$$

W. König proved that,

- If $\{U_t\}$ is strongly continuous on H^p then $g = \frac{\partial m_t}{\partial t} |_{t=0}$ exists, it is analytic on \mathbf{D} , and $\{m_t\}$ is given in terms of g as above.

- If $\{m_t\}$ is given in terms of some g and $\sup_{z \in \mathbb{D}} \operatorname{Re}(g(z)) < \infty$ then $\{U_t\}$ is strongly continuous on H^p and the infinitesimal generator is given by

$$\Delta(f)(z) = G(z)f'(z) + g(z)f(z),$$

where G is the generator of $\{\phi_t\}$.

Thus Δ is a perturbation, by the multiplication operator $M_g(f) = gf$, which may be unbounded, of the generator Γ of the unweighted semigroup.

In particular if $m_t = w \circ \phi_t / w$ then the generator is

$$\Delta(f) = Gf' + (w'G/w)f = (G/w)(wf)'$$

and we have the following formal identity

$$M_w \circ \Delta = \Gamma \circ M_w,$$

where M_w is multiplication by w . This intertwining property for generators is in fact inherited from a corresponding property of the semigroups. Indeed comparing the unweighted semigroup $\{T_t\}$ with $\{S_t\}$ we see that

$$M_w \circ S_t = T_t \circ M_w \quad \text{for all } t \geq 0.$$

Examples and applications

1. Let $\phi_t(z) = e^{-t}z + 1 - e^{-t}$, and $w(z) = 1 - z$, then

$$S_t(f)(z) = \frac{w(\phi_t(z))}{w(z)} f(\phi_t(z)) = e^{-t} f(\phi_t(z)).$$

The generator is

$$\Delta(f)(z) = (1 - z)f'(z) - f(z) = ((1 - z)f(z))'.$$

It can be shown that

$$\sigma(\Delta) = \{z : \operatorname{Re} z \leq -1 + \frac{1}{p}\}.$$

Thus $0 \in \rho(\Delta)$ exactly when $p > 1$. Denoting by \mathcal{A} the resolvent $R(0, \Delta)$ we have

$$\mathcal{A}(f)(z) = \frac{1}{z - 1} \int_1^z f(\zeta) d\zeta$$

If $p = 1$ then $0 \in \sigma(\Delta)$ and this is an immediate proof that \mathcal{A} is not bounded on H^1 . With the standard basis for H^p the matrix for \mathcal{A} is

$$\begin{pmatrix} 1, & 1/2, & 1/3, & \cdots \\ 0, & 1/2, & 1/3, & \cdots \\ 0, & 0, & 1/3, & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix},$$

the transpose of the Cesàro matrix. On H^2 therefore \mathcal{A} and the Cesàro operator \mathcal{C} are Hilbert

space adjoints. This information was used by C. Cowen to show that \mathcal{C} is a subnormal operator on H^2 .

Norm estimates for the semigroup can be used to obtain for $1 < p < \infty$,

$$\|\mathcal{A}\|_{H^p} = \frac{p}{p-1},$$

$$\sigma(\mathcal{A}) = \left\{ z : \left| z - \frac{p}{2(p-1)} \right| \leq \frac{p}{2(p-1)} \right\}.$$

2. Let $\phi_t(z) = \frac{e^{-t}z}{(e^{-t}-1)z+1}$ and

$$S_t(f)(z) = \frac{\phi_t(z)}{z} f(\phi_t(z)).$$

The generator is given by

$$\begin{aligned} \Delta(f)(z) &= -z(1-z)f'(z) - (1-z)f(z) \\ &= -(1-z)(zf(z))'. \end{aligned}$$

The spectrum of Δ is

$$\sigma(\Delta) = \{z : \operatorname{Re}(z) \leq -1/p\}.$$

Thus $0 \in \rho(\Delta)$ and we find $R(0, \Delta) = \mathcal{C}$ the Cesàro operator,

$$\begin{aligned} \mathcal{C}(f)(z) &= \frac{1}{z} \int_0^z f(\zeta) \frac{1}{1-\zeta} d\zeta \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^n a_k \right) z^n. \end{aligned}$$

It takes some heavy calculation to obtain

$$\|S_t\|_{H^p} = e^{-t/p}$$

(valid however only for $p \geq 2$). From these and the Hille-Yosida theorem one obtains

$$\|\mathcal{C}\|_{H^p} = p, \quad p \geq 2.$$

and

$$\sigma(\mathcal{C}) = \left\{ z : \left| z - \frac{p}{2} \right| \leq \frac{p}{2} \right\},$$

for all $p \geq 1$.

It is interesting to note that \mathcal{C} can be also obtained from the semigroup

$$\phi_t(z) = 1 - (1-z)e^{-t} = h^{-1}(e^{-t}h(z))$$

with $h(z) = \log(1/(1-z))$. Indeed we find

$$R(1, \Gamma)(f)(z) = \frac{1}{\log(1/(1-z))} \int_0^z f(\zeta) \frac{1}{1-\zeta} d\zeta.$$

so

$$\begin{aligned} \mathcal{C}(f)(z) &= \frac{1}{z} \log\left(\frac{1}{1-z}\right) \int_0^1 f(1 - (1-z)^s) ds \\ &= \frac{1}{z} \log\left(\frac{1}{1-z}\right) \int_0^1 f(\phi_s(z)) ds. \end{aligned}$$

Concluding remarks

Semigroups of composition operators form a large class of examples of operator semigroups in Banach spaces. Their study gives insight about single composition operators (even for those which do not participate in a semigroup) as well as for some classical integration operators.

The univalent function h associated to a semigroup $\{\phi_t\}$, is, in principle, determined by the boundary of the region Ω onto which it maps the disc. Thus the properties of $\{T_t\}$ are determined by this geometry of the boundary, and would be interesting to find this relation.

Questions that remain, are related to strong continuity (especially for weighted semigroups), spectra (especially the case of DW point =1), stability, orbits, dynamic and asymptotic behavior e.t.c.