Univalent Geodesics, Alternate Löwner-Kufarev Equation and Virasoro Algebra

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(joint work with Irina Markina)

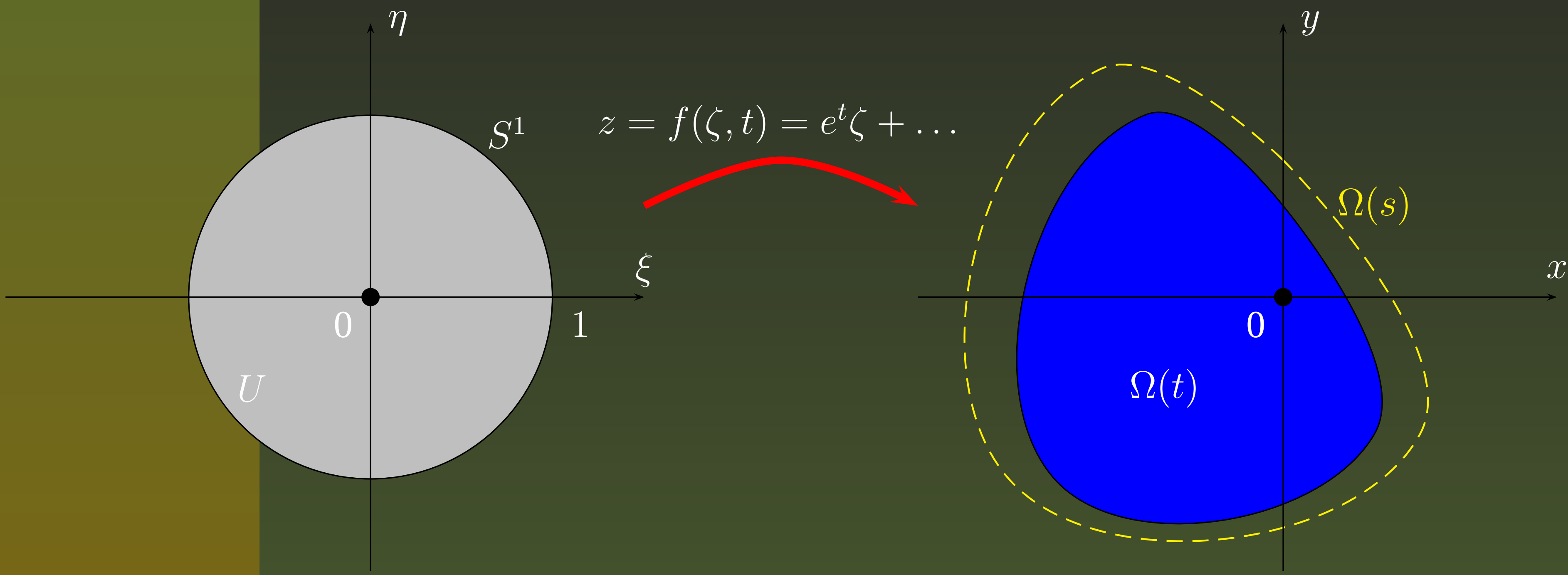
Löwner Theory

- Charles Loewner (Karel Löwner)

  1893 Bohemia, Czech Republic– 1968 Stanford, USA

K. Löwner, *Untersuchungen über schlichte konforme Abbildungen des Einheitskreises. I*, Math. Ann. **89** (1923), no. 1-2, 103–121.

Löwner Subordination



- $\Omega(t) \subset \Omega(s)$ as $t < s$.

Löwner Equation

Given a subordination chain of domains $\Omega(t)$ defined for $t \in [0, T)$, there exists an analytic regular function

$$p(\zeta, t) = 1 + p_1(t)\zeta + p_2(t)\zeta^2 + \dots, \quad \zeta \in U = \{\zeta : |\zeta| < 1\},$$

such that Re $p(\zeta, t) > 0$ and

$$\frac{\partial f(\zeta, t)}{\partial t} = \zeta \frac{\partial f(\zeta, t)}{\partial \zeta} p(\zeta, t),$$

for $\zeta \in U$ and for almost all $t \in [0, T)$, T may be ∞

Löwner Equation

- Christian Pommerenke

born December 17, 1933 København, Danmark

Ch. Pommerenke, *Über die Subordination analytischer Funktionen*, J. Reine Angew. Math. **218** (1965), 159–173.

P. P. Kufarev

Pavel Parfenievich Kufarev (1909–1968) was born and died in Tomsk. His life was always linked with the Tomsk State University where he studied (1927–1932), was appointed as docent (1935), professor (1944), State Honor in Sciences (1968). His main achievements are in the theory of Univalent Functions where he generalized in several ways the famous Löwner parametric method.

P. P. Kufarev

P.P. Kufarev, *On one-parameter families of analytic functions*, Rec. Math. [Mat. Sbornik] N.S. **13(55)** (1943), 87–118.

Kufarev proved the existence of the derivative $\frac{\partial f}{\partial t}$ almost everywhere in $t \in [0, T)$ and ζ from the Carathéodory kernel of $\{\Omega(t)\}_{t\in[0,T)}$. So we refer to *Löwner-Kufarev equation*.

Characteristic Equation

- Inverse problem: given a domain $\Omega(0) \equiv \Omega_0$ (therefore, $f(\zeta, 0) \equiv f_0(\zeta)$), and a regular function $p(\zeta, t)$, Re $p > 0$, let us solve the Löwner-Kufarev equation.
- Problem: Does the solution $f(\zeta, t)$ represent a subordination chain of simply connected univalent domains?

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- $\dfrac{dt}{ds} = 1, \quad \dfrac{d\zeta}{ds} = -\zeta p(\zeta, t), \quad \dfrac{df}{ds} = 0$, with the initial conditions $t(0) = 0$, $\zeta(0) = z$, $f(\zeta, 0) = f_0(\zeta)$

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- The Löwner-Kufarev equation in ordinary derivatives for a function $\zeta = w(z, t)$

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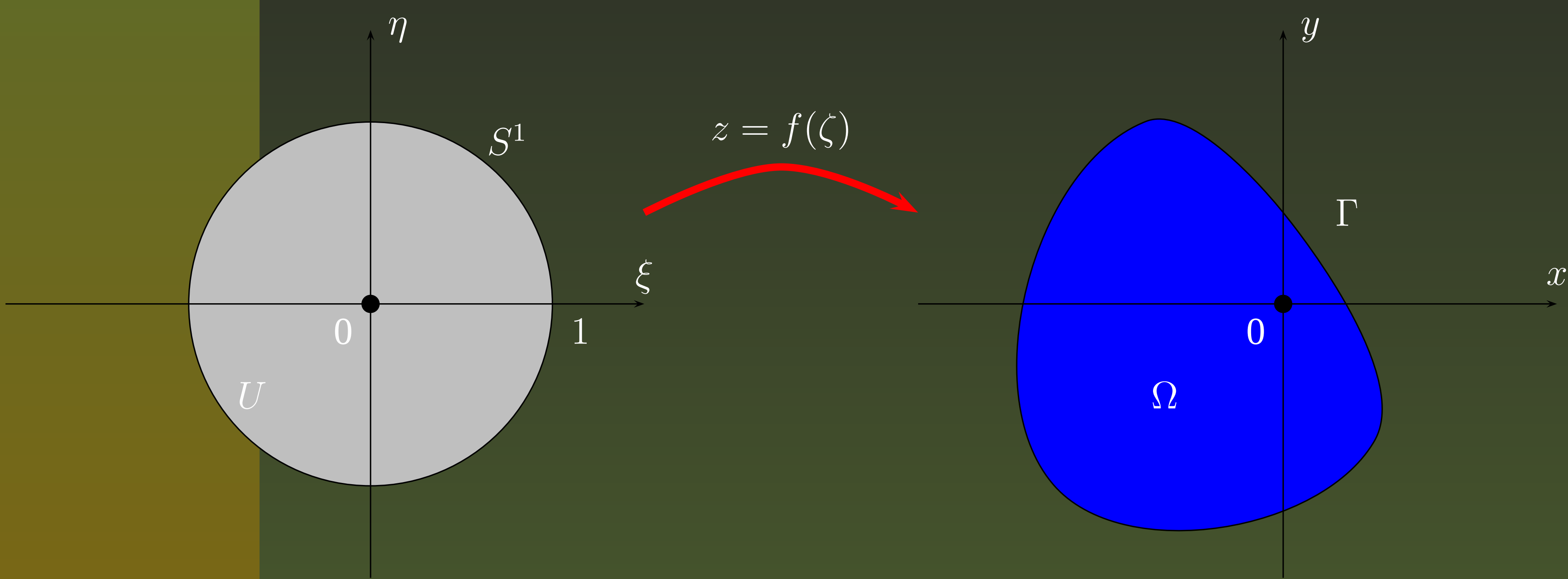
- The solution is given by $f_0(w^{-1}(\zeta, t))$. Problem: ζ is not defined in U.

Löwner-Kufarev Representation



The answer to the above problem is found in the Löwner-Kufarev representation.

Löwner-Kufarev Representation

- Any univalent function $f : U \to \Omega$, $f(z) = z + c_1 z^2 + \ldots$ can be represented as a limit

$$f(z) = \lim_{t \to \infty} e^t w(z, t).$$

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- The function $\zeta = w(z,t)$,

$$w(z,t) = e^{-t} z \left(1 + \sum_{n=1}^{\infty} c_n(t) z^n\right),$$

satisfies

$$\frac{dw}{dt} = -wp(w,t),$$

with the initial condition $w(z,0) = z$.

Existence and univalence

- PDE Löwner-Kufarev equation $\dot{f} = zf'p(z,t)$, Re $p(z,t) > 0$ with the initial condition $f(z,0) = f_0(z)$ has a unique univalent solution all the time $t \in [0,\infty)$
  if $f_0(z) = \lim_{t\to\infty} e^t w(z,t)$, where the function $\zeta = w(z,t)$,
  $$w(z,t) = e^{-t}z\left(1 + \sum_{n=1}^{\infty} c_n(t)z^n\right),$$
  satisfies
  $$\frac{dw}{dt} = -wp(w,t),$$
  with the initial condition $w(z,0) = z$.

Reverse construction

- If we already know the solution $f(\zeta, t) = e^t\zeta + \dots$ to Löwner-Kufarev PDE $\dot{f} = zf'p(z,t)$, Re $p(z,t) > 0$ with the initial condition $f(z,0) = f_0(z)$ for $t \in [0,\infty)$, then the solution to the corresponding Löwner-Kufarev ODE is constructed as

$$w(z,t) = f^{-1}(f_0(z), t) = e^{-t}z + \dots$$

- no problem with extension.

Algebraic structures

- Interesting phenomenon: Similar algebraic structures appear in different theories in physics and mathematics.
- Our interest– Virasoro algebra.

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- The Virasoro Algebra plays a fundamental role in Conformal Field Theory,
- In non-linear equations, the Virasoro algebra is intrinsically related to the KdV canonical structure,
- The Virasoro Algebra is an important object in mathematics as an example of infinite-dimensional Lie-Fréchet algebra.

When and where?

- Miguel Ángel Virasoro
  born in Argentina in 1940
  Argentinean physicist, former director of ICTP in Trieste,
  Professor, Universita' di Roma 'La Sapienza'

When and where?

1970 M.A.Virasoro: *Subsidiary conditions and ghosts in dual-resonance models*.- Phys. Rev. D, **1** (1970), 2933–2936.

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- However the Virasoro Algebra was studied earlier in 1968.

When and where?

1970 M.A.Virasoro: *Subsidiary conditions and ghosts in dual-resonance models*.- Phys. Rev. D, **1** (1970), 2933–2936.

1968 I.M.Gel'fand, D.B.Fuchs: *Cohomology of the Lie algebra of vector fields on the circle*.- Functional Anal. Appl. **2** (1968), no. 4, 342–343.

Formal Definition

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$$L_n = -z^{n+1}\frac{\partial}{\partial z}, \quad n \in \mathbb{Z}.$$

- The Lie-Poisson bracket of two Killing fields is

$$[L_m, L_n] = (m-n)L_{m+n}.$$

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The constant c is the *central charge* and it is a constant of the theory.

Conformal Field Theory

- In Conformal Field Theory Virasoro Algebra *Vir* appears as an infinite dimensional algebra generated by the coefficients of the Laurent expansion of the analytic component T of the momentum-energy tensor, Virasoro generators.

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- In Conformal Field Theory Virasoro Algebra *Vir* appears as an infinite dimensional algebra generated by the coefficients of the Laurent expansion of the analytic component T of the momentum-energy tensor, Virasoro generators.
- The corresponding Virasoro-Bott group *vir* appears as the space of reparametrization of a closed string.

KdV hierarchy

- Phase space (field variables) $u(e^{ix}, t)$ so that $S^1 \times \mathbb{R}$ is our space-time. Simplifying $u \to u(x,t)$, where the new u is a 2π periodic smooth function.

KdV hierarchy

- KdV equation $u_t = 6uu_x + u_{xxx}$ has an infinite number of conserved quantities (first integrals) $I_k[u]$, e.g.,

$$I_{-1} = \int u dx, \;\; I_0 = \int u^2 dx, \;\; I_1 = \int (\frac{1}{2}(u'^2) + u^3) dx, \ldots$$

$$\ldots, I = \int \text{polynomial } (\tfrac{d}{dx}, \cdot u) dx.$$

which are all in involution (I_{-1}- mass, I_0- energy).

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- Hierarchy is constructed as

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- Lax reformulation

$$\dot{L} = [L, A_n], \quad L = -\partial^2 + u, \quad A_n = 4i(L^{\frac{2n+1}{2}})_{\geq 0}.$$

KdV via Virasoro

- Virasoro generators

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- Define

$$u = \frac{6}{c}\sum_{n\in\mathbb{Z}} L_n e^{-inx} - \frac{1}{4}$$

KdV via Virasoro

- Virasoro generators

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- Then, using $\delta(x) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} e^{inx}$, we obtain

$$[u(x), u(y)] = \frac{6\pi}{c}(-\delta'''(x-y) + 4u(x)\delta'(x-y) + 2u'\delta(x-y)).$$

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- Taking $I_0 = \frac{1}{2}\int_0^{2\pi} u^2 dx$, we obtain

$$\dot{u} = \frac{c}{6\pi}[u, I_0] = -u''' + 6uu'.$$

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- $\phi(\theta + 2\pi) = \phi(\theta)$;
- The commutator $[\phi_1, \phi_2] = \phi_1\phi_2' - \phi_1'\phi_2$.

Some obstacles

- Finite dimension: Lie algebra-Lie group correspondence.
- Infinite dimension: Lie-Banach, Lie-Fréchet.
- The Lie algebra Vect S^1 can not be lifted to the Lie-Fréchet group Diff S^1.
- What are coordinated on Vect S^1?

Kirillov's construction

- Kirillov proposed to consider the homogeneous space Diff S^1/S^1, and the Lie algebra Vect $S^1/const =$ Vect $_0S^1$ (i.e., $\int_0^{2\pi}\phi d\theta = 0$).

Complex Structure

- A complex structure for Vect $_0 S^1$

$$\phi(\theta) = \sum_{n=1}^{\infty} a_n \cos n\theta + b_n \sin n\theta,$$

$$J[\phi](\theta) = \sum_{n=1}^{\infty} -a_n \sin n\theta + b_n \cos n\theta,$$

Complexification Vect $_0 S^1 \otimes \mathbb{C} =$ Vect $_0^+ S^1 \oplus$ Vect $_0^- S^1$;
projections:

$$\phi \to v := \frac{1}{2}(\phi \mp iJ[\phi]) = \sum_{n=1}^{\infty} (a_n \mp i b_n) e^{in\theta} \in \text{Vect}_0^{\pm} S^1.$$

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- $[v_n, v_m]_{Vir} = (m-n)v_{n+m} + \omega(v_n, v_m)$.

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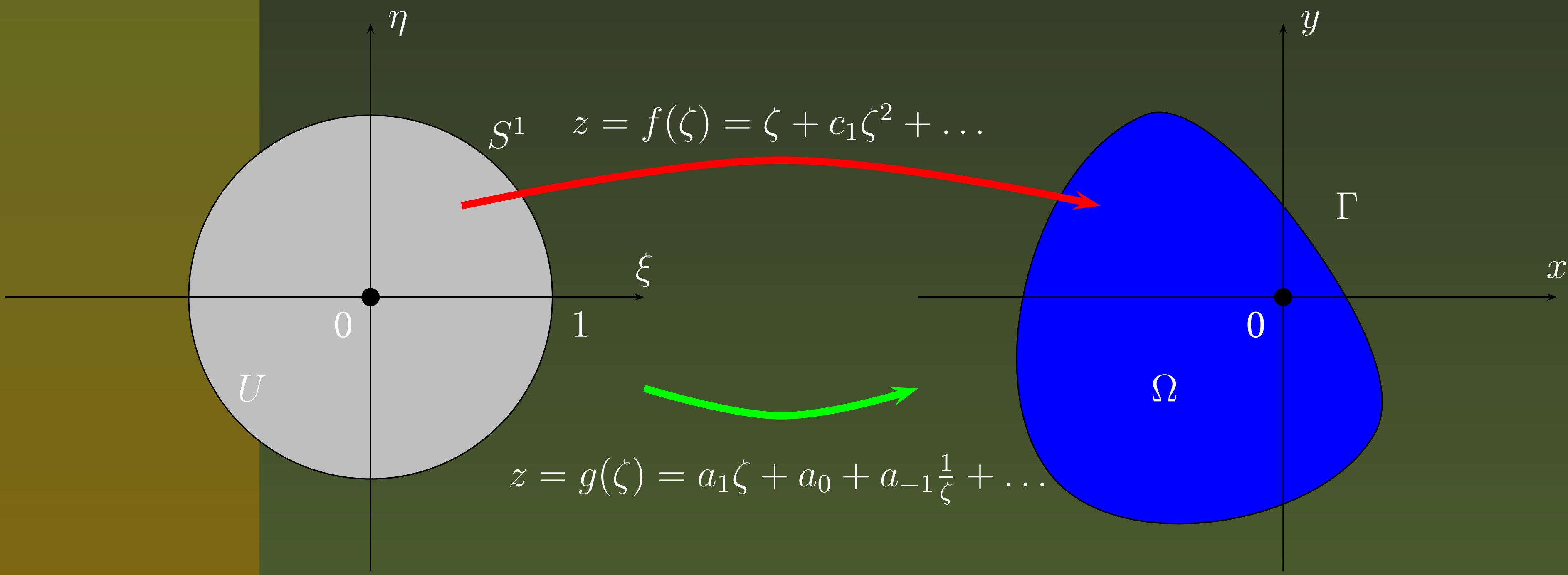

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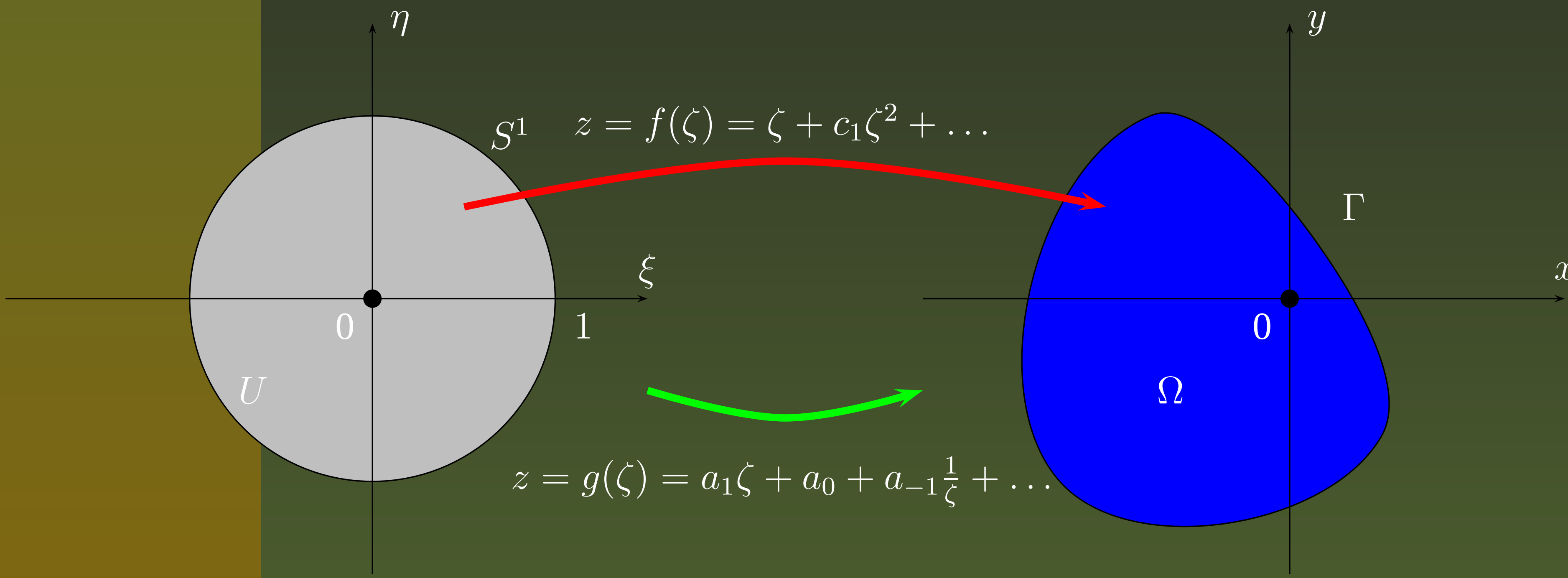

- $\gamma = f^{-1} \circ g|_{S^1} \in \text{Diff } S^1/S^1$, $\quad f \in \mathbf{S} \leftrightarrows \gamma \in \text{Diff } S^1/S^1$.
- We identify the space of smooth Jordan curves and Diff S^1/S^1.

Schaeffer and Spencer Variation

$$\delta_\phi f(z) = \frac{f^2(\zeta)}{2\pi} \int_{S^1} \left(\frac{w f'(w)}{f(w)} \right)^2 \frac{\phi(w)dw}{w(f(w) - f(z))},$$

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- δ_ϕ transfers the complex structure J from Vect ${}_0S^1$ to $T\mathbf{S}$: $J(\delta_\phi) := \delta_{J(\phi)}$.
- Complexification $T\mathbf{S} = T\mathbf{S}^+ \oplus T\mathbf{S}^-$: $\delta_\phi \mp iJ(\delta_\phi) \in T\mathbf{S}^\pm$

$$v = \phi \mp iJ(\phi) = \sum_{n=1}^{\infty} (a_n \mp ib_n) e^{in\theta}.$$

Kirillov's vector fields

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- Taking $v_k = -i\zeta^k$, $k = 1, 2, \ldots$ for $T\mathbf{S}^+$, we obtain

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- Commutators:

$$[L_m, L_n] = (n - m)L_{m+n}, \quad [L_{-n}, L_n] = 2nL_0,$$

where $L_0(f)(z) = zf'(z) - f(z)$. L_0 corresponds to rotation.

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- It is easily seen from

$$f_\varepsilon(z) = e^{-i\varepsilon} f(e^{i\varepsilon} z) = f(z) + i\varepsilon(zf'(z) - f(z)) + o(\varepsilon).$$

Virasoro algebra

- Virasoro algebra (complex) is a central extension

$$[L_m, L_n]_{Vir} = (m-n)L_{m+n} + \frac{c}{12}n(n^2-1)\delta_{n,-m},$$

$c \in \mathbb{C}$.

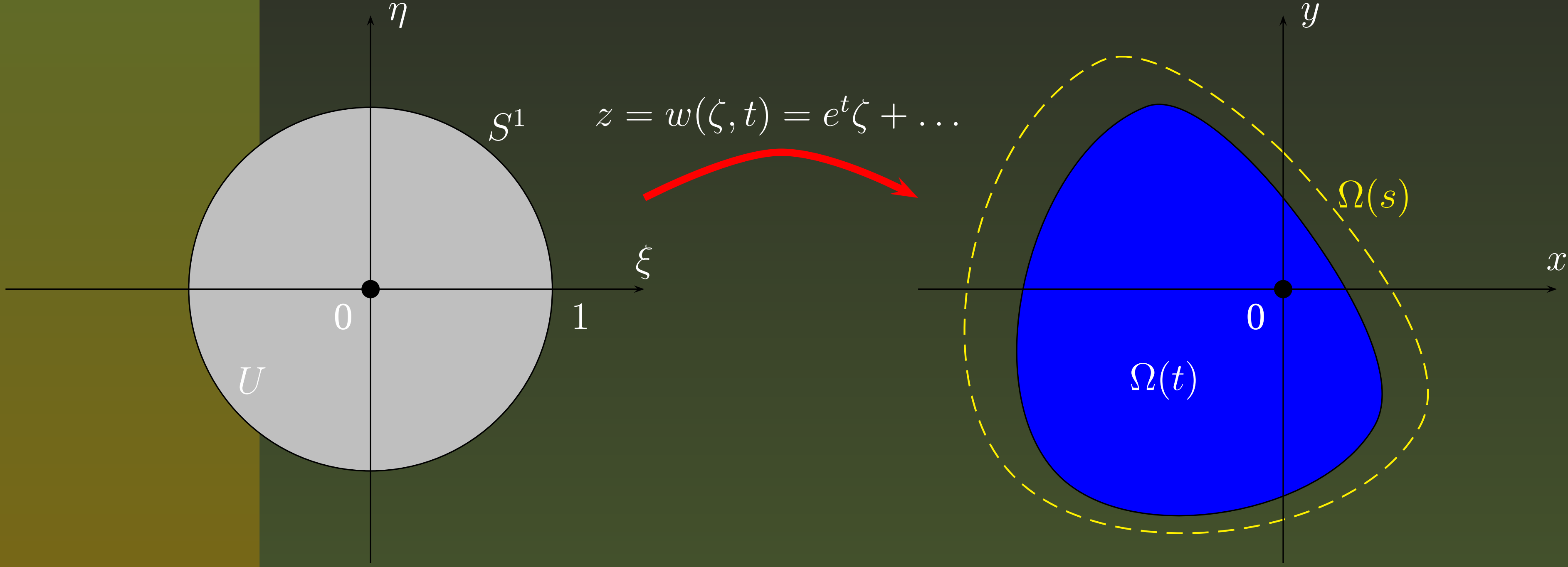
We concentrate our attention on $T\mathbf{S}^+$.

- In affine coordinates we get Kirillov's operators:

$$L_j = \partial_j + \sum_{k=1}^{\infty}(k+1)c_k\partial_{j+k} \quad \partial_j = \partial/\partial c_j,$$

$j = 1, 2, \ldots$

Algebraic structure of L-K



- $\Omega(t) \subset \Omega(s)$ as $t < s$.
- The Löwner-Kufarev equation

$$\dot{w}(\zeta, t) = \zeta w'(\zeta, t) p(\zeta, t), \quad \text{Re } p(\zeta, t) > 0, \quad |\zeta| < 1.$$

Curve in coefficient body

- $f(z,t) = e^{-t}w(z,t) = z(1 + \sum_{n=1}^{\infty} c_n z^n)$;
- smooth curve: $(c_1(t), \dots, c_n(t), \dots)$;
- tangent vector: $\dot{c}_1\partial_1 + \dots + \dot{c}_n\partial_n + \dots$, $\partial_n = \frac{\partial}{\partial c_n}$;
- recalculation in a new basis $\{L_1, \dots, L_n, \dots\}$

$$\dot{c}_1\partial_1 + \dots + \dot{c}_n\partial_n + \dots = u_1 L_1 + \dots u_n L_n + \dots;$$

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$$\dot{w}(\zeta, t) = \zeta w'(\zeta, t) p(\zeta, t), \quad \mathrm{Re}\, p(\zeta, t) > 0, \quad |\zeta| < 1.$$

- $f(z, t) = e^{-t} w(z, t) = z(1 + \sum_{n=1}^{\infty} c_n z^n)$;
- compare with the Löwner equation

$$\dot{f} = (\dot{c}_1 \partial_1 + \dots + \dot{c}_n \partial_n + \dots) f = z f' p(z, t) - f =,$$

$$= (L_0 + u_1 L_1 + \dots u_n L_n + \dots) f,$$

where $p(z, t) = 1 + u_1 z + \dots + u_n z^n + \dots$.

- $L_n = \partial_n + \sum_{k=1}^{\infty} (k+1) c_k \partial_{n+k}, \qquad L_n f(z) = z^{n+1} f'(z).$

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- $L_n = \partial_n + \sum_{k=1}^{\infty} (k+1) c_k \partial_{n+k}, \qquad L_n f(z) = z^{n+1} f'(z).$
- What is L_0?

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$$\dot{f} = (\dot{c}_1\partial_1 + \dots + \dot{c}_n\partial_n + \dots)f = zf'p(z,t) - f =,$$

$$= (L_0 + u_1L_1 + \dots u_nL_n + \dots)f,$$

where $p(z,t) = 1 + u_1 z + \dots + u_n z^n + \dots$.

- $L_n = \partial_n + \sum_{k=1}^{\infty}(k+1)c_k\partial_{n+k}, \qquad L_n f(z) = z^{n+1}f'(z).$
- The answer is in the normalization $f(z,t) = z(1 + \sum_{n=1}^{\infty} c_n(t)z^n)$

Dynamics

- The dynamics without normalization and subordination;
- $f(z,t) = a_0(t)z + a_1(t)z^2 + \dots;$
- tangent vector: $\dot{a}_0\partial_0 + \dots + \dot{a}_n\partial_n + \dots;$
- recalculation in the new basis

$$\dot{a}_0\partial_0 + \dots + \dot{a}_n\partial_n + \dots = u_0L_0 + \dots u_nL_n + \dots;$$

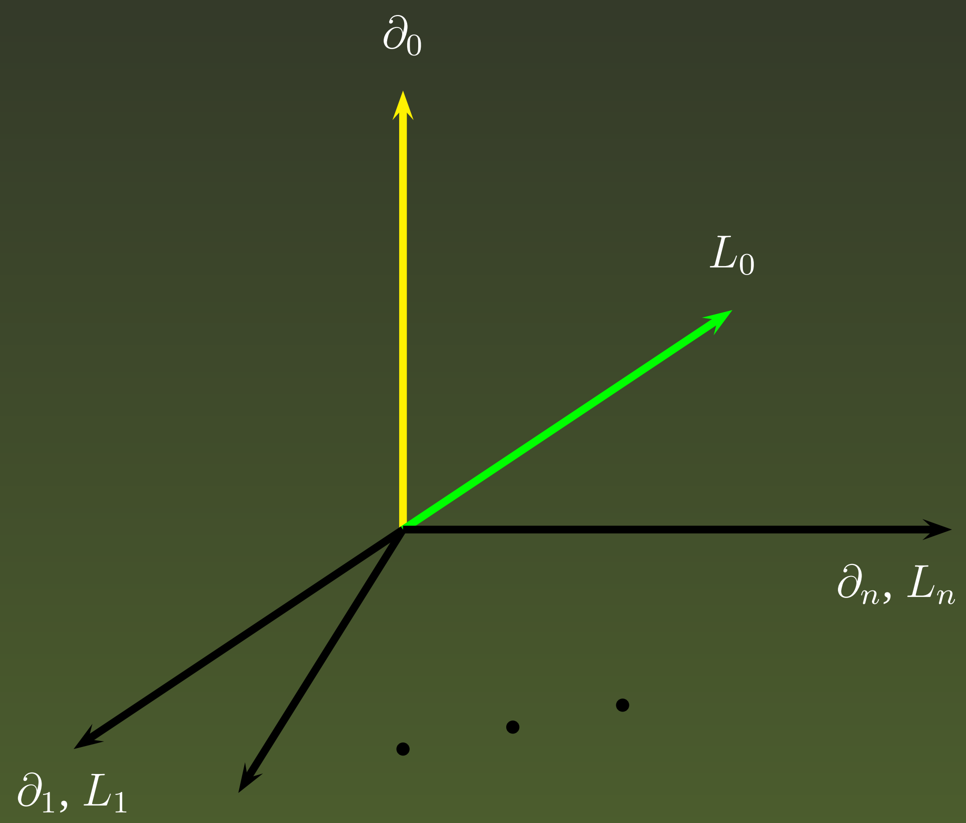
- $L_n = a_0\partial_n + 2a_1\partial_{n+1} + \dots,\ \partial_n = \frac{\partial}{\partial a_n},\ n = 0, 1, \dots;$
- $\partial_n f = z^{n+1},\ L_n(f) = z^{n+1}f',$

$$\dot{f} = \dot{c}_1\partial_1 + \dots + \dot{c}_n\partial_n + \dots = zf'p(z,t) = u_0L_0 + u_1L_1 + \dots u_nL_n + \dots$$

where $p(z,t) = u_0 + u_1z + \dots + u_nz^n + \dots$.

Projections

The dynamics is performed in the space of co-dimension 1:



We consider two projections: w.r.t. ∂_0 and w.r.t. L_0.

Analytic form of projections ∂_0

- $F_1(z,t) = \frac{1}{a_0} f(z,t) = z + \frac{a_1}{a_0} z^2 + \dots$

$$\dot{F}_1 = zF_1' p(z,t) - \frac{\dot{a}_0}{a_0} F_1,$$

where $u_0 = \frac{\dot{a}_0}{a_0}$.

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- $$\dot{c}_1 \partial_1 + \dots + \dot{c}_n \partial_n + \dots = \hat{L}_0 + u_1 \hat{L}_1 + \dots + u_n \hat{L}_n + \dots$$

  where $\hat{L}_0 F_1 = u_0(zF_1' - F_1)$, $\hat{L}_k F_1 = z^{k+1} F_1'$ (in particular, $a_0 = e^t \Rightarrow$ Löwner-Kufarev), $c_k = \frac{a_k}{a_0}$, $\partial_k = \frac{\partial}{\partial c_k}$.

- The Löwner PDE is an analytic form of the recalculation of the tangent vector from the basis ∂_n to the basis L_n and the projection w.r.t. $a_0 = e^t$.

Analytic form of projections L_0

- $F_2(z,t) = f(\frac{1}{a_0}z,t) = z + \frac{a_1}{a_0^2}z^2 + \dots$

$$\dot{F}_2 = zF_2'p(\frac{z}{a_0},t) - \frac{\dot{a}_0}{a_0}zF_2',$$

where $u_0 = \frac{\dot{a}_0}{a_0}$.

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- $$\dot{c}_1\partial_1 + \dots + \dot{c}_n\partial_n + \dots = u_1\tilde{L}_1 + \dots + u_n\tilde{L}_n + \dots$$

where $\tilde{L}_k F_1 = z^{k+1}F_1'$, $c_k = \frac{a_k}{a_0^{k+1}}$, $\partial_k = \frac{\partial}{\partial c_k}$.

Commutators

- In all cases:
- $L_n(f) = z^{n+1} f'$ for $n = 1, 2, 3, \dots$
- $L_n(f) = z^{n+1} f'$ for $n = 0, 1, 2, 3, \dots$
- $L_n(f) = z^{n+1} f'$ for $n = 1, 2, 3, \dots, \quad L_0 = zf' - f,$

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TheWitt commutator relation is

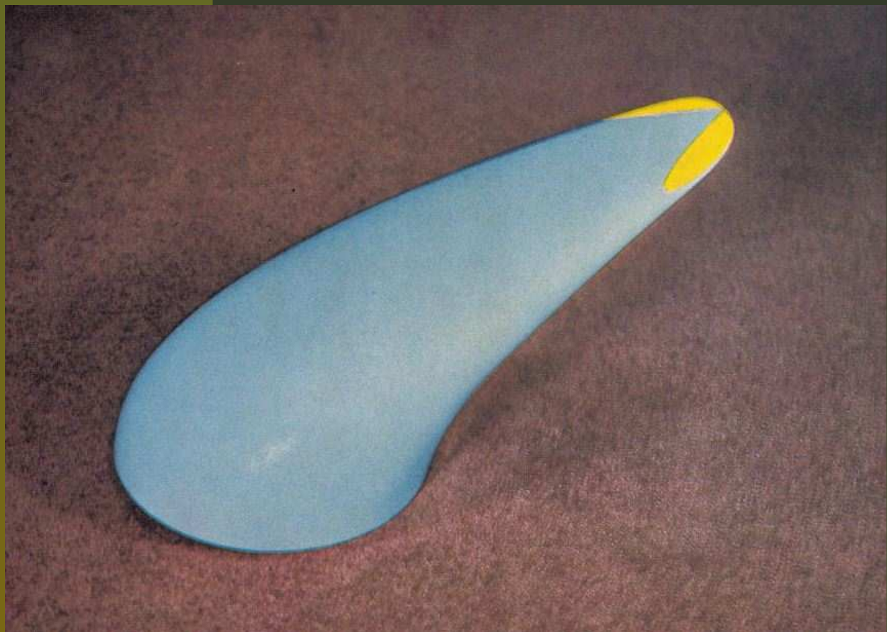
- $\{L_n, L_m\} = (m - n)L_{n+m}.$

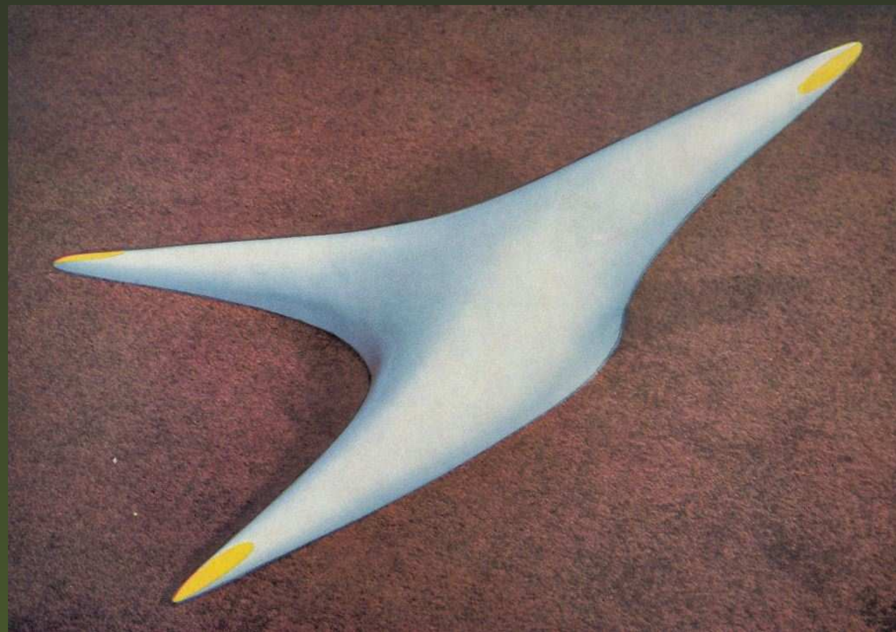
Coefficient bodies

- Curves within the coefficient bodies $\mathcal{M}_n = (c_1, c_2, \ldots, c_n)$, $f(z,t) = z(1 + \sum_{n=1}^{\infty} c_n(t) z^n)$, $f \in S$.
- For $n = 1$, $\mathcal{M}_1 = \{|c_1| \leq 2\}$;
- For $n = 2$ non-trivial
- A. C. Schaeffer, D. C. Spencer, *Coefficient Regions for Schlicht Functions* , American Mathematical Society Colloquium Publications, Vol. 35. American Mathematical Society, New York, 1950.

Body $\mathcal{M}_2$

Crossections





$(c_1, \text{Re } c_2, \text{Im } c_2)$ $(\text{Re } c_1, \text{Im } c_1, c_2)$

Curves in $\mathcal{M}_n$

- The operators L_j restricted onto $\mathcal{M}_n$ give truncated vector fields
$$L_j = \partial_j + \sum_{k=1}^{n-j}(k+1)c_k\partial_{j+k};$$
- Let $c(t) = \big(c_1(t), \dots, c_n(t)\big)$ be a smooth trajectory in $\mathcal{M}^n$;
- $\dot{c}(t) = \dot{c}_1(t)\partial_1 + \ldots + \dot{c}_n(t)\partial_n = u_1 L_1 + u_2 L_3 + \ldots + u_n L_n.$
- L_j are differential 1-st order operators. The operator $L = \sum |L_k|^2$ is elliptic.

Hamiltonian

- Turning to co-vectors $L_k \longrightarrow l_k$ we write the Hamiltonian defined on the co-tangent bundle

$$H(c,\bar{c},\psi,\bar{\psi}) = \sum_{k=1}^{n} |l_k|^2,$$

where

$$l_k = \bar{\psi}_k + \sum_{j=1}^{n-k}(j+1)c_j\bar{\psi}_{k+j}.$$

The co-vectors $\bar{\psi}_k$ correspond to ∂_k

Hamiltonian system

$$
\begin{aligned}
\dot{c}_1 &= \frac{\partial H}{\partial \bar{\psi}_1} = \bar{l}_1 \\
\dots &= \dots\dots\dots\dots \\
\dot{c}_n &= \frac{\partial H}{\partial \bar{\psi}_n} = \bar{l}_n + \sum_{j=1}^{n-1} (j+1) c_j \bar{l}_{n-j} \\
\dot{\bar{\psi}}_p &= -\frac{\partial H}{\partial c_p} = -(p+1) \sum_{k=1}^{n-p} l_k \bar{\psi}_{k+p} \\
\dots &= \dots\dots\dots\dots \\
\dot{\bar{\psi}}_n &= -\frac{\partial H}{\partial c_n} = 0.
\end{aligned}
$$

Geodesics, Lagrangian

- $\dot{c}_1(t)\partial_1 + \ldots + \dot{c}_n(t)\partial_n = u_1 L_1 + u_2 L_3 + \ldots + u_n L_n$

Results:

- $l_k = \bar{u}_k,\, k = 1, 2 \ldots, n;$
- $\dot{u}_k = \sum_{j=1}^{n-k} (j-k)\bar{u}_j u_{j+k},$
- $H = \sum_{k=1}^{n} |l_k|^2$ =const, along geodesics,
- Lagrangian $L = (\dot{c}, \bar{\psi}) - H = \frac{1}{2}\sum_{k=1}^{n} |u_k|^2$,
- turning to ∞ dimension and to the Löwner-Kufarev equation, $L = \|p\|^2_{H^2}$.

Some conclusions

- The Löwner-Kufarev PDE can be considered as an algebraic recalculation of basis.
- From this point of view, the driving term $p(z,t)$ does need to be of Re $p > 0$.
- Alternate Löwner equation.
- One of the reasons: Brownian motion on Jordan curves.

Brownian motion on Jordan curves

- We consider the canonical Brownian motion on the group of diffeomorphisms of the unit circle and on the space of Jordan curves.

Brownian motion on Jordan curves

- The regularized canonical Brownian motion on Diff S^1 is a stochastic flow on S^1 associated to the Itô stochastic differential equation

$$dg^r_{x,t} = d\zeta^r_{x,t}(g^r_{x,t}),$$

$$\zeta^r_{x,t}(\theta) = \sum_{n=1}^{\infty} \frac{r^n}{\sqrt{n^3 - n}} (x_{2n}(t)\cos n\theta - x_{2n-1}(t)\sin n\theta),$$

where $\{x_k\}$ is a sequence of independent real-valued Brownian motions and $r \in (0,1)$ and the series for $\zeta^r_{x,t}(\theta)$ is a Gaussian trigonometric series.

Brownian motion on Jordan curves

- Kunita's theory of stochastic flows asserts that the mapping $\theta \to g^r_{x,t}(\theta)$ is a C^∞ diffeomorphism and the limit $\lim_{r \to 1^-} g^r_{x,t} = g_{x,t}$ exists uniformly in θ. The random homeomorphism $g_{x,t}$ is called *canonical Brownian motion* on Diff S^1. (Airault, Fang, Malliavin, Ren, Zhang). ;

Brownian motion on Jordan curves

- The canonical Brownian motion given on Diff S^1 can be defined also on the space of C^∞-smooth Jordan curves by conformal welding.
- No subordination, alternate behavior.

L-K representation again

- Any univalent function $f : U \to \Omega$, $f(z) = z + c_1 z^2 + \dots$ ($f \in \mathbf{S}$) can be represented as a limit

$$f(z) = \lim_{t\to\infty} e^t w(z,t).$$

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- The function $\zeta = w(z,t)$,

$$w(z,t) = e^{-t} z \left(1 + \sum_{n=1}^{\infty} c_n(t) z^n \right),$$

satisfies

$$\frac{dw}{dt} = -wp(w,t),$$

with the initial condition $w(z,0) = z$.

L-K representation again

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$$f(z) = \lim_{t \to \infty} e^t w(z,t).$$

- If $p(z,t)$ is analytic in $z \in U$ and smooth in $\hat{U} = U \cup S^1$, then $w(z,t)$ analytic, univalent in $z \in U$ and smooth in $\hat{U} = U \cup S^1$.

Hamiltonian system for ODE

- The Hamiltonian system

$$\frac{d(e^t w(z,t))}{dt} = e^t w(1 - p(w,t)) = \frac{\delta H}{\delta \overline{\psi}} = \{H, \overline{\psi}\}.$$

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$$H = \int\limits_{z\in S^1} e^t w(z,t)(1-p(w(z,t),t))\bar{\psi}(z,t)\frac{dz}{iz}, \quad \psi(z) = \sum_{1}^{\infty} \psi_n z^n,$$

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$\psi(z,t)$ is holomorphic in $\hat{U}$,

- $$\frac{d\bar{\psi}}{dt} = -(1 - p(w,t) - wp'(w,t))\bar{\psi} = \frac{-\delta H}{\delta(e^t w)} = \{H, e^t w\}.$$

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$\psi(z,t)$ is holomorphic in $\hat{U}$,

- The conservative quantity is $L(z) = e^t w'(z,t)\bar{\psi}(z,t)$. All equations are given on the unit circle $|z| = 1$.

Conservative Quantities

- Considering $L(z)_{<0} = [e^t w'(z,t)\bar{\psi}(z,t)]_{<0}$ we get the Kirillov's fields

$$L_j = \partial_j + \sum_{k=1}^{\infty}(k+1)c_k\partial_{j+k},$$

where $\bar{\psi}_k$ is the co-vector for ∂_k.

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- The operators L_n are defined on the co-tangent space to the class $\tilde{\mathbf{S}}$ (smooth on S^1).

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- The Witt commutator relation is $\{L_n, L_m\} = (m-n)L_{n+m}$.

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where $\bar{\psi}_k$ is the co-vector for ∂_k.

- Considering $[e^t w'(z,t)\bar{\psi}(z,t)]_{\geq}$ we get analogous fields $\mathcal{L}_n$ but the mixed commutators $[L_k, \mathcal{L}_n]$ do not satisfy Witt relation.

Conclusion

- Conservative quantities for the Löwner-Kufarev ODE with a smooth driving term represent holomorphic coordinates of the Virasoro algebra.
- These conservative quantities are the Kirillov's operators in the representation of the Virasoro Algebra.
- Alternate Löwner-Kufarev PDE is the correspondence $T\mathbf{S}^+ \leftrightarrow T\mathbf{S}$ of old and new complex structures on the Virasoro algebra.

Open questions

- How can the antiholomorphic part of the Virasoro algebra be realized within Löwner-Kufarev theory?
- Brownian motion on Jordan curves: the dimension of the intersection of the curves locally?
- Authentic Lagrangian and Hamiltonian formulation of Löwner evolution?
- The same question for the Laplacian growth?

The End