INDAM Workshop on Holomorphic Iteration, Semigroups, and Loewner Chains

Rome, 9-12 September 2008

Commuting semigroups of holomorphic mappings

M.Elin

(joint work with

M.Levenshtein, S.Reich, D.Shoikhet)

Notations

$$\Delta := \{z \in \mathbf{C} : |z| < 1\}$$

$$\overline{\Delta} := \{z \in \mathbf{C} : |z| \le 1\}$$

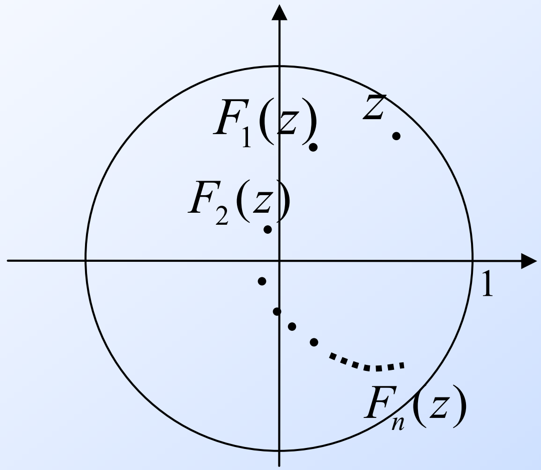
$$\partial\Delta = \{z \in \mathbf{C} : |z| = 1\}$$

$\mathrm{Hol}(\Delta, D)$ - the set of all holomorphic functions on Δ which map Δ into a set $D \subset \mathbf{C}$

$\mathrm{Hol}(\Delta) := \mathrm{Hol}(\Delta, \Delta)$ - the set of all holomorphic self-mappings of Δ

Iterations

Let $F \in Hol(\Delta)$ be a holomorphic self-mapping of the unit disk



$$F_0(z) = z$$

$$F_1(z) = F(z)$$

$$F_2(z) = F(F(z))$$

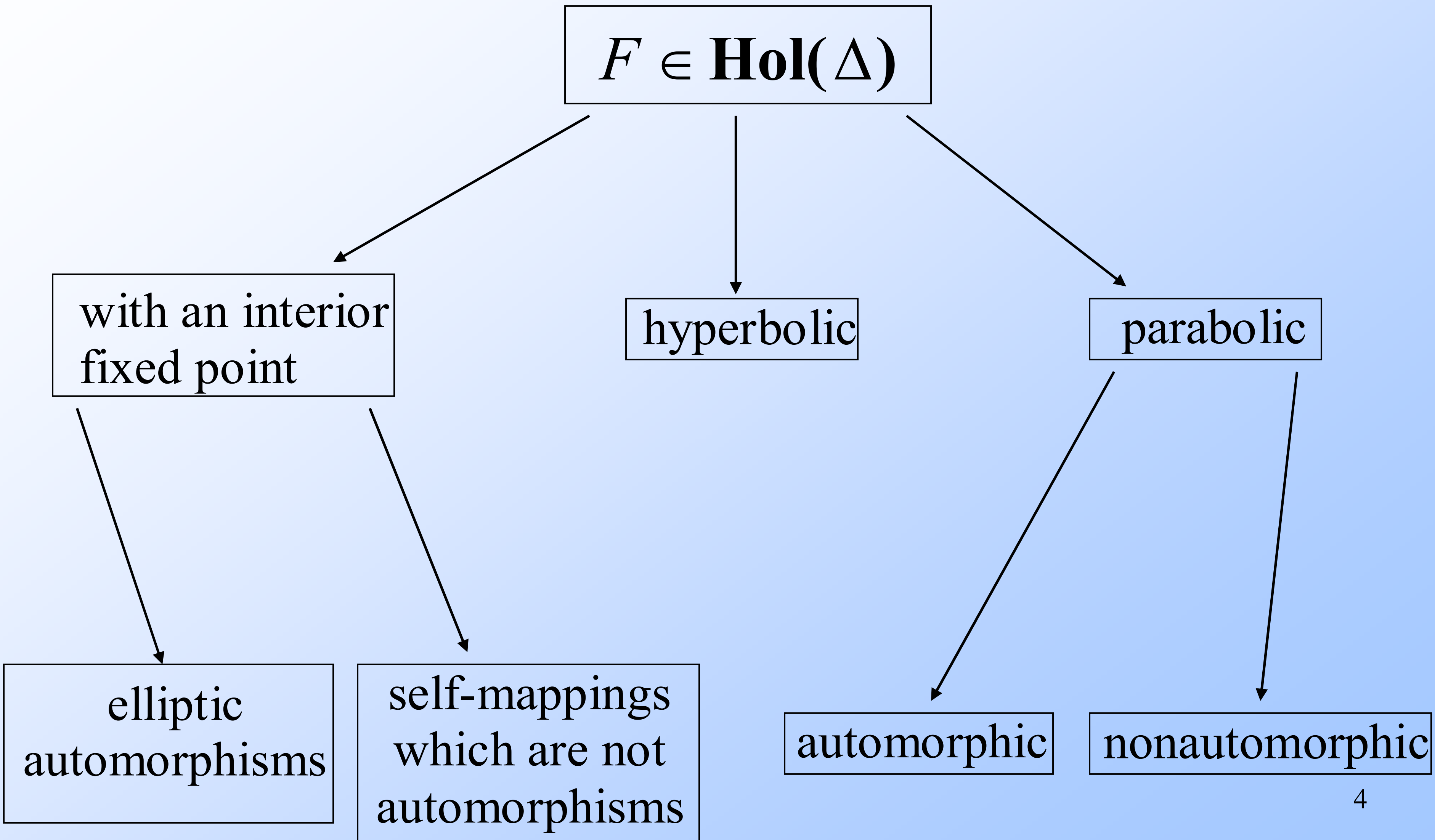
$$\ldots\ldots\ldots$$

$$F_n(z) = F(F_{n-1}(z))$$

If $F, G \in Hol(\Delta)$ then

$$F \circ G = G \circ F \quad \Rightarrow \quad F_n \circ G_k = G_k \circ F_n$$

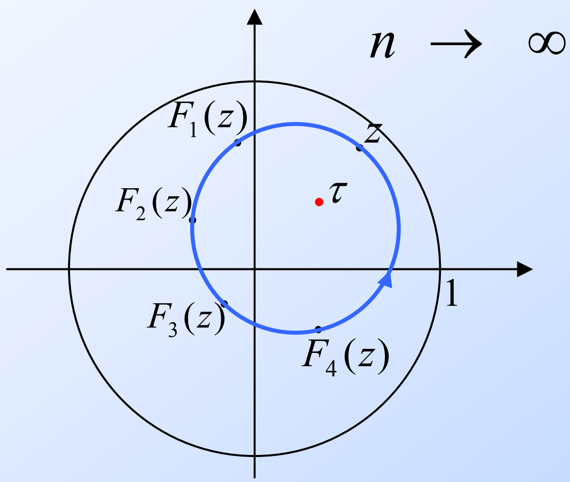
Classification



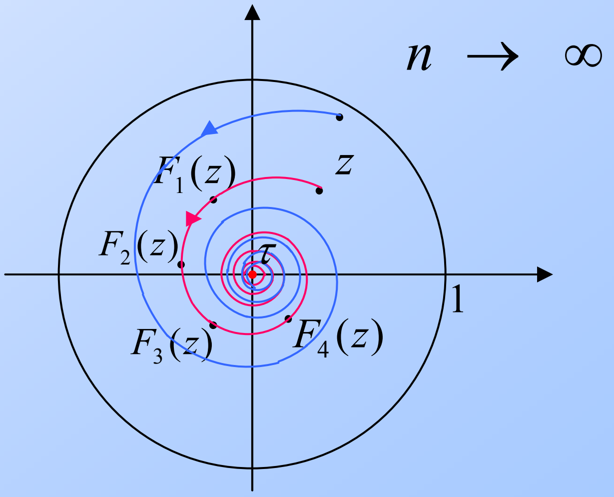
Self-mappings with an interior fixed point

Fixed point: $F(\tau)=\tau, \ \ \tau\in\Delta$

F is an elliptic automorphism of Δ



F isn't an elliptic automorphism of Δ

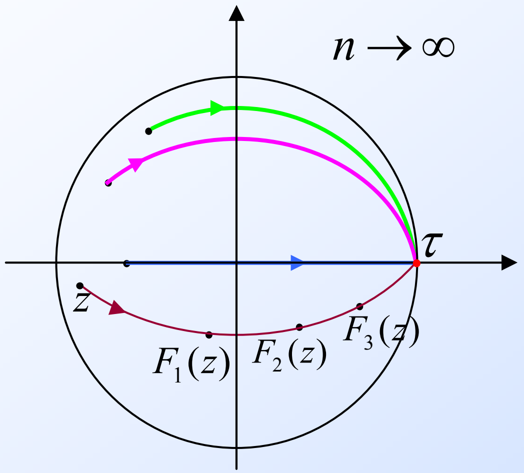


$$\lim_{n\to\infty} F_n(z)=\tau$$

Self-mappings with no interior fixed point

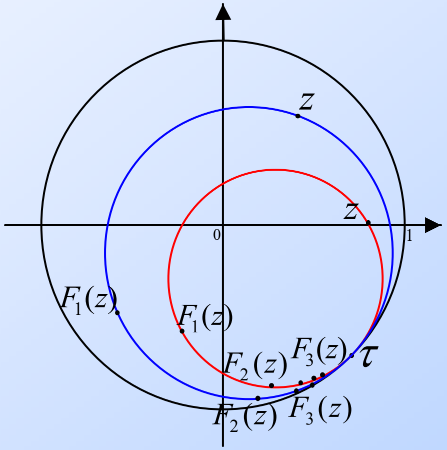
There is a boundary point $\tau \in \partial\Delta$ such that $\lim_{n\to\infty} F_n(z) = \tau$

$$0 < F'(\tau) \le 1$$



Hyperbolic type:

$0 < F'(\tau) < 1$



Parabolic type:

$F'(\tau) = 1$

Commuting self-mappings

$F, G \in \mathbf{Hol}(\Delta)$ which are different from the identity mapping

$F \circ G = G \circ F \quad \Rightarrow \quad F, G$ are of the same type:

parabolic, hyperbolic, or with an interior fixed point.

M. H. Heins (1941)

If $F \in \mathrm{Aut}_{hyp}(\Delta)$, then $G \in \mathrm{Aut}_{hyp}(\Delta)$

If F is of hyperbolic type, but $F \notin \mathrm{Aut}(\Delta) \overset{?}{\to} G$ is of parabolic type

C. C. Cowen (1984)

If $F,\ G$ are not automorphisms, then they are of the same type

F is of hyp.type, $F \notin \mathbf{Aut}(\Delta) \Rightarrow G \notin \mathbf{Aut}_{par}(\Delta)$

Continuous Semigroups

A family $S = \{F_t\}_{t\geq 0} \subset Hol(\Delta)$ *is called*
a one-parameter continuous semigroup *if*

$(i)\quad F_{t+s}(z) = F_t(F_s(z)) \quad for \quad all \quad t, s \in [0,\infty) \quad and \quad z \in D$

$(ii)\quad \lim_{t\to 0^+} F_t(z) = z \quad for \quad all \quad z \in D$

The local continuity condition (ii) implies the differentiability of S with respect to the parameter $t \geq 0$ (Berkson&Porta (1978)). The limit

$$\lim_{t\to 0^+} \frac{z - F_t(z)}{t} := f(z), \quad z \in \Delta$$

defines a holomorphic function on **Δ,** which is called the ***(infinitesimal) generator of S.***

Generators and Semigroups

There is a unique point $\tau \in \overline{\Delta}$ such that

$$f(z) = (z-\tau)(1-z\overline{\tau})\,p(z), \quad z \in \Delta,$$

with $\mathbf{Re}\, p(z) \geq 0$ for all $z \in \Delta$

The point τ is the Denjoy-Wolff point (attractive fixed point) of the semigroup generated by f.

$$F_t'(\tau) = e^{-f'(\tau)t}$$

$$F_t''(\tau) = \begin{cases} -\dfrac{f''(\tau)}{f'(\tau)} e^{-f'(\tau)t}\left(1 - e^{-f'(\tau)t}\right), & f'(\tau) \neq 0 \\ -f''(\tau)t, & f'(\tau) = 0 \end{cases}$$

Commuting Semigroups

$$F_t \circ F_s = F_s \circ F_t, \quad \forall s,t \geq 0$$

Let $S_1 = \{F_t\}_{t\geq 0}, \quad S_2 = \{G_t\}_{t\geq 0}$ be two continuous semigroups

We say that ***two semigroups commute*** *if*

$$F_t \circ G_s = G_s \circ F_t, \quad \forall s,t \geq 0$$

Suppose that $F_1 \circ G_1 = G_1 \circ F_1$

Problem:

$$F_1 \circ G_1 = G_1 \circ F_1 \quad \overset{?}{\not\Rightarrow} \quad F_t \circ G_s = G_s \circ F_t, \quad \forall s,t \geq 0$$

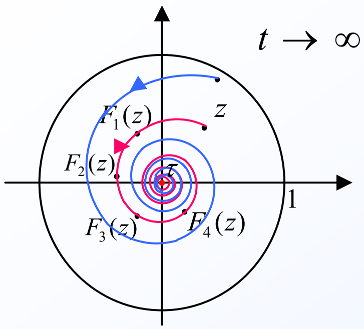
$F_1 \circ G_1 = G_1 \circ F_1$ and F_1, G_1 are different from the identity $\Rightarrow$ $S_1 = \{F_t\}_{t \geq 0}$, $S_2 = \{G_t\}_{t \geq 0}$ are of the same type

---

$\{F_t\}_{t \geq 0} \in \mathbf{Hol}(\Delta)$

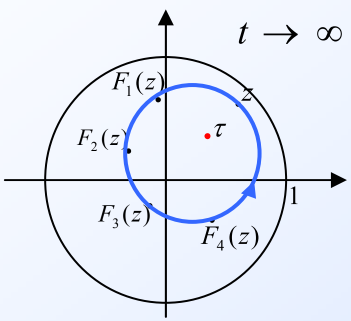
- with an interior fixed point
  - elliptic automorphisms
  - self-mappings which are not automorphisms
- hyperbolic
- parabolic
  - automorphic
  - nonautomorphic

Semigroups with an interior fixed point



$S_1 = \{F_t\}_{t\ge 0}, S_2 = \{G_t\}_{t\ge 0}$ semigroups of self-mappings which are not automorphisms

$$F_1 \circ G_1 = G_1 \circ F_1 \quad \Rightarrow \quad F_t \circ G_s = G_s \circ F_t, \quad \forall s,t \ge 0$$



$S_1 = \{F_t\}_{t\ge 0}$ a semigroup of elliptic automorphisms with a common fixed point at $\tau \in \Delta$

S_1 and $S_2 = \{G_t\}_{t\ge 0}$ are commuting iff

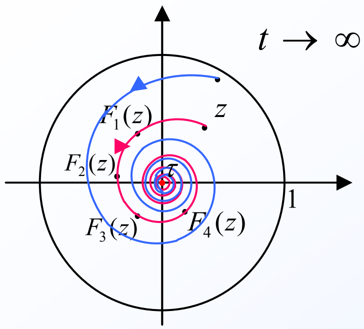
$$G_t(z) = m_\tau\left(e^{-at} \cdot m_\tau(z)\right), \quad a \in \mathbf{C}, \quad m_\tau(z) = \frac{\tau - z}{1 - \bar{\tau} z}.$$

<u>Corollary 1</u>. $G_t(z) = m_{\tau_1}\left(e^{i\varphi t} \cdot m_{\tau_1}(z)\right)$ for some $\varphi \in \mathbf{R}$

$S_1 = \{F_t\}_{t\ge 0}, S_2 = \{G_t\}_{t\ge 0}$ semigroups of elliptic automorphisms:

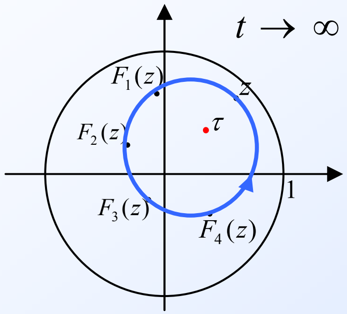
$$F_1 \circ G_1 = G_1 \circ F_1 \quad \Rightarrow \quad F_t \circ G_s = G_s \circ F_t, \quad \forall s,t \ge 0$$

Semigroups with an interior fixed point



$S_1 = \{F_t\}_{t\ge 0}, S_2 = \{G_t\}_{t\ge 0}$ semigroups of self-mappings which are not automorphisms

$$F_1 \circ G_1 = G_1 \circ F_1 \quad \Rightarrow \quad F_t \circ G_s = G_s \circ F_t, \quad \forall s,t \ge 0$$



$S_1 = \{F_t\}_{t\ge 0}$ - a semigroup of elliptic automorphisms with a common fixed point at $\tau \in \Delta$

$S_2 = \{G_t\}_{t\ge 0}$ - a semigroup of self-mappings

$$F_1 \circ G_1 = G_1 \circ F_1 \quad \Rightarrow \quad F_1 \circ G_s = G_s \circ F_1, \quad \forall s \ge 0$$

Corollary 2. $F_t(z) = m_\tau\left(e^{i\varphi t} \cdot m_\tau(z)\right)$ for some $\varphi \in \mathbf{R}$

If $\frac{\varphi}{\pi}$ is an irrational number, then

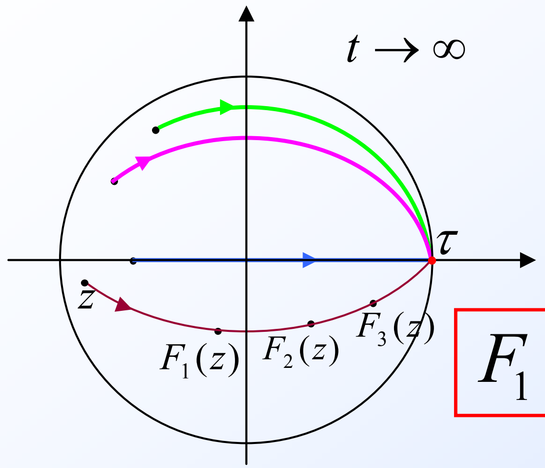
$F_1 \circ G_1 = G_1 \circ F_1 \Rightarrow F_1 \circ G_t = G_t \circ F_1, \ \forall t \ge 0$, but S_1, S_2 do not commute

Example. $F_t(z) = e^{i\varphi t} z$

$\varphi = \frac{2\pi}{n}, \quad g(z) = zp(z^n), \ \operatorname{Re} p(z) \ge 0 \quad \forall z \in \Delta$

$$F_1 \circ G_1 = G_1 \circ F_1 \Rightarrow F_t \circ G_s = G_s \circ F_t, \ \forall s,t \ge 0$$

Semigroups of hyperbolic type



Hyperbolic type:

$$0 < F_t'(\tau) < 1$$

$$F_1 \circ G_1 = G_1 \circ F_1 \quad \Rightarrow \quad F_t \circ G_s = G_s \circ F_t, \quad \forall s,t \geq 0$$

$\{F_t\}_{t\geq 0} \subset \mathrm{Hol}(\Delta)$

- with an interior fixed point
  - elliptic automorphisms
  - $S_1 \subset \mathbf{Aut}(\Delta)$, $S_2 \not\subset \mathbf{Aut}(\Delta)$
  - self-mappings which are not automorphisms
- hyperbolic

Semigroups of parabolic type

parabolic

automorphic

$$\lim_{n\to\infty} \rho(F_n(z), F_{n+1}(z)) > 0$$

nonautomorphic

$$\lim_{n\to\infty} \rho(F_n(z), F_{n+1}(z)) = 0$$

$z, w \in \Delta$

The Poincaré hyperbolic metric:

$$\rho(z,w) := \frac{1}{2}\log\frac{1+|m_w(z)|}{1-|m_w(z)|}, \quad m_w(z) = \frac{w-z}{1-\overline{w}z}$$

$\{\rho(F_n(z), F_{n+1}(z))\}$ - a non-increasing sequence

Semigroups of parabolic type

Let $S_1 = \{F_t\}_{t\geq 0}, S_2 = \{G_t\}_{t\geq 0}$ be semigroups of parabolic nonautomorphic type. Then

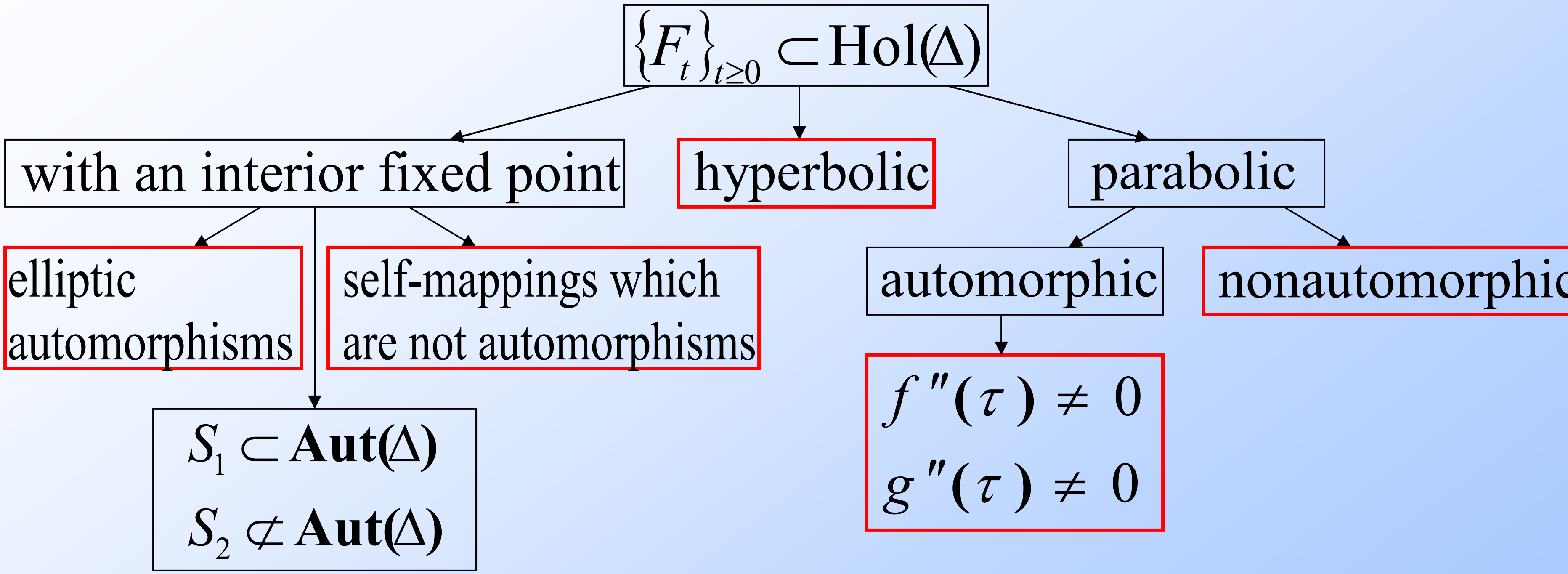
$$F_1 \circ G_1 = G_1 \circ F_1 \quad \Rightarrow \quad F_t \circ G_s = G_s \circ F_t, \quad \forall s,t \geq 0$$

Let at least one of the semigroups S_1, S_2 is of automorphic type.

If $$\begin{cases} f, g \subset C^2(\tau) \\ f''(\tau) \neq 0, \quad g''(\tau) \neq 0 \end{cases}$$

then $$F_1 \circ G_1 = G_1 \circ F_1 \Rightarrow F_t \circ G_s = G_s \circ F_t, \quad \forall s,t \geq 0$$

Summary



$\{F_t\}_{t\geq 0} \subset \mathrm{Hol}(\Delta)$
with an interior fixed point
hyperbolic
parabolic
elliptic automorphisms
self-mappings which are not automorphisms
$S_1 \subset \mathbf{Aut}(\Delta)$
$S_2 \not\subset \mathbf{Aut}(\Delta)$
automorphic
nonautomorphic
$f''(\tau) \neq 0$
$g''(\tau) \neq 0$

Open questions

$$F_1 \circ G_1 = G_1 \circ F_1 \quad \overset{?}{\Rightarrow} \quad F_t \circ G_s = G_s \circ F_t\,, \quad \forall t \ge 0, s \ge 0$$

$\lim\limits_{z\to\tau} \dfrac{f(z)}{\tau - z} \ne 0, \qquad \lim\limits_{z\to\tau} \dfrac{g(z)}{\tau - z} \ne 0$ - semigroups of hyperbolic type

$\lim\limits_{z\to\tau} \dfrac{f(z)}{(\tau - z)^2} \ne 0, \qquad \lim\limits_{z\to\tau} \dfrac{g(z)}{(\tau - z)^2} \ne 0$ -semigroups of parabolic type

$\lim\limits_{z\to\tau} \dfrac{f(z)}{(\tau - z)^3} \ne 0, \qquad \lim\limits_{z\to\tau} \dfrac{g(z)}{(\tau - z)^3} \ne 0$ -semigroups of parabolic nonautomorphic type

$$\lim_{z\to\tau} \frac{f(z)}{(\tau - z)^{\alpha+1}} \ne 0, \qquad \lim_{z\to\tau} \frac{g(z)}{(\tau - z)^{\alpha+1}} \ne 0 \quad \text{for some} \quad \alpha \in [0,2]$$

Open questions

Conjecture

$$\lim_{z\to\tau}\frac{f(z)}{(\tau-z)^{\alpha+1}}\neq 0, \qquad \lim_{z\to\tau}\frac{g(z)}{(\tau-z)^{\alpha+1}}\neq 0 \quad \text{for} \quad \text{some} \quad \alpha\in[0,2]$$

$$\Downarrow$$

$$F_1\circ G_1=G_1\circ F_1 \quad \Rightarrow \quad F_t\circ G_s=G_s\circ F_t, \quad \forall t\geq 0, s\geq 0$$

Question Do there exist two semigroups of parabolic type such that

$$F_1\circ G_1=G_1\circ F_1, \quad \text{but for some} \quad t_0\geq 0, s_0\geq 0$$

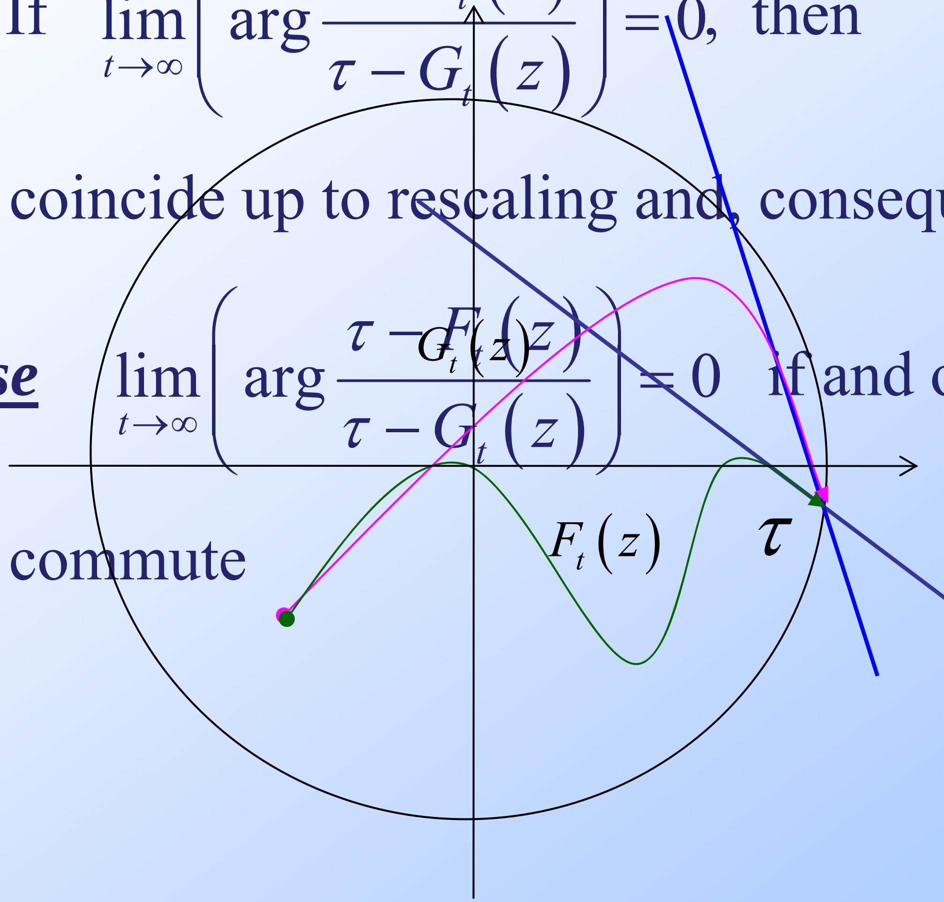
$$F_t\circ G_s\neq G_s\circ F_t\,?$$

Angular Asymptotic Characteristics

F.Jacobzon, S.Reich, D. Shoikhet &M.E.

- ***Dilation case*** If $\lim_{t\to\infty}\left(\arg\dfrac{\tau - F_t(z)}{\tau - G_t(z)}\right) = 0$, then the semigroups coincide up to rescaling and, consequently, commute

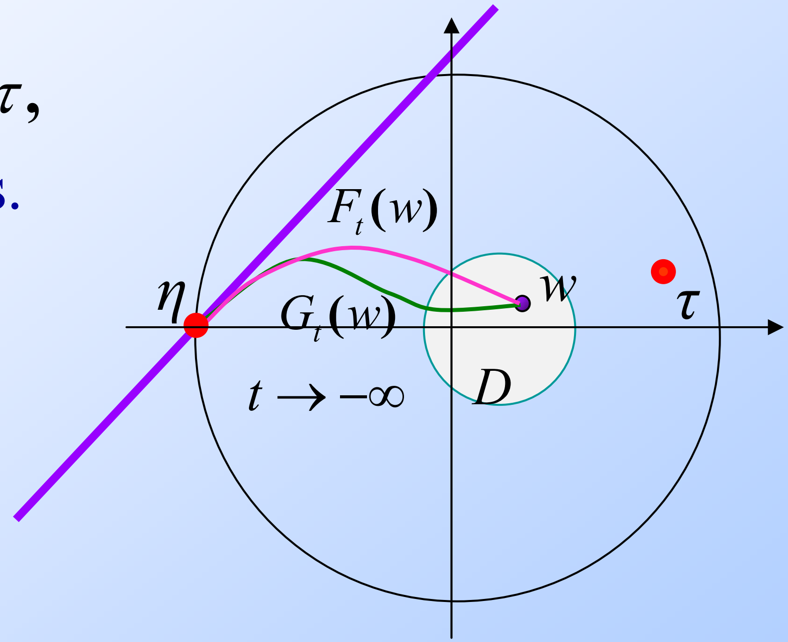
- ***Hyperbolic case*** $\lim_{t\to\infty}\left(\arg\dfrac{\tau - F_t(z)}{\tau - G_t(z)}\right) = 0$ if and only if the semigroups commute



$$\lim_{t\to\infty}\left(\arg\frac{\tau - F_t(z)}{\tau - G_t(z)}\right) = 0$$

Repelling points

Suppose that there is a point $\eta \in \partial\Delta, \eta \neq \tau$, such that $f(\eta) = 0$ and $\angle \lim\limits_{z\to\eta} \dfrac{f(z)}{z-\eta}$ exists.



Then for some point $w \in \Delta$, the orbit $F_t(w) \in \Delta$ can be extended for all real t and $\lim\limits_{t\to-\infty} F_t(w) = \eta$.

If there is a nonempty open set $D \subset \Delta$, where

$$\lim_{t\to-\infty} \arg \frac{\eta - F_t(w)}{\eta - G_t(w)} = 0 \quad \text{for all} \quad w \in D$$

then the semigroups coincide up to rescaling and, consequently, commute.

Thank you for your attention!