# *Sub-Riemannian view on $SU(2)$ and semigroup of its sub-Laplacian*

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# Definitions

Let $M^n$ be a differentiable manifold, $TM^n$ be a tangent bundle, and $\langle\cdot,\cdot\rangle$ be a positively definite metric on $TM^n$

$(M^n, TM^n, \langle\cdot,\cdot\rangle_{TM^n})$ is a Riemannian manifold

# Definitions

Let $M^n$ be a differentiable manifold, $TM^n$ be a tangent bundle, and $\langle\cdot,\cdot\rangle$ be a positively definite metric on $TM^n$

$$(M^n, TM^n, \langle\cdot,\cdot\rangle_{TM^n}) \quad \text{is a Riemannian manifold}$$

Take a manifold $M^n$, a distribution of $k$-dimensional planes $\mathcal{D}^k \subset TM^n$, $k < n$, and a positively definite metric $\langle\cdot,\cdot\rangle$ on $\mathcal{D}^k$

$$(M^n, \mathcal{D}^k, \langle\cdot,\cdot\rangle_{\mathcal{D}^k}) \quad \text{is a sub-Riemannian manifold}$$

# Examples

- Parallel parking,

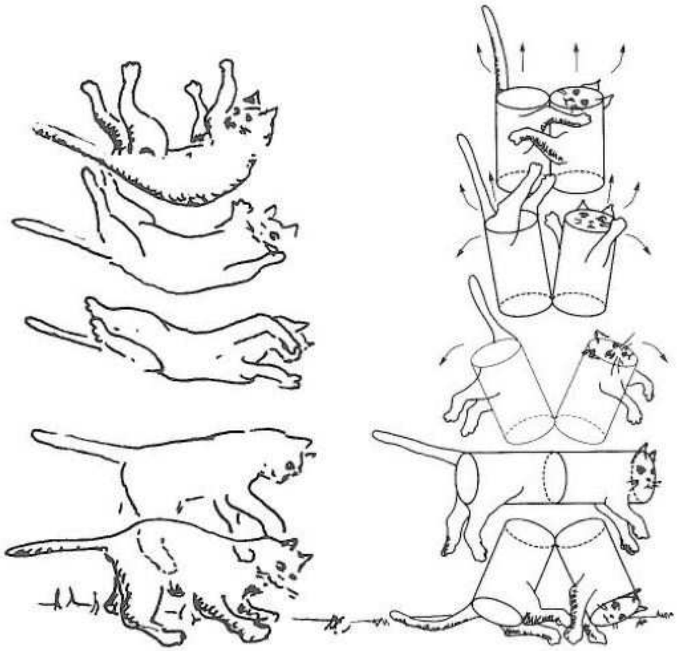
# Examples

- Rolling ball without slipping and twisting,

Sub-Riemannin manifold:
$(\mathbb{R}^5, \mathbb{R}^2,$ Euclidean metric on the plane$)$

# Examples

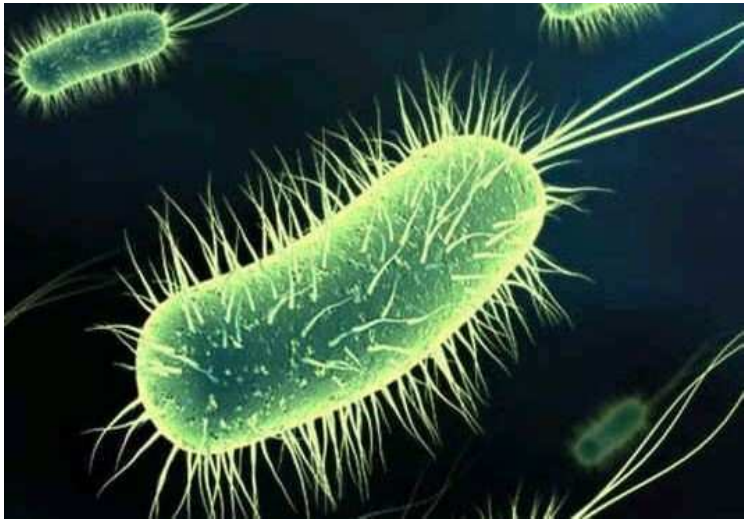
- falling cat



How does a cat falling in mid-air with no angular momentum, spin itself around and right itself?

# Examples

- swimming



Microorganisms live in an environment dominated by viscous drag and Brownian motion. How a cyclic motion of a body results to propel it forward?

# Geometry of principal bundles

Let $M$ be a configuration space of a mechanical system with a kinetic energy $T$ and a potential energy $U$. If a group $G$ acts on $M$: $G \circlearrowleft M$ freely and leaves the energies invariant, then the quotient map

$$h: \; M \;\rightarrow\; M/G = Q$$

gives the configuration space $M$ the structure of the principal $G$-bundle. We pullback the metric $\rho_Q$ to $M$:

$$h^*\rho_Q(X,Y) = \rho_Q(h_*X, h_*Y)$$

and it gives the sub-Riemannian structure to $M$: $(M, TQ, h^*\rho_Q)$.

# Geometry of principal bundles

To the shortest curve $\gamma$ in the configuration space $M$ corresponds the shortest curve $c$ in the base space $Q$ (which is the projection under $h: \ M \to Q$).

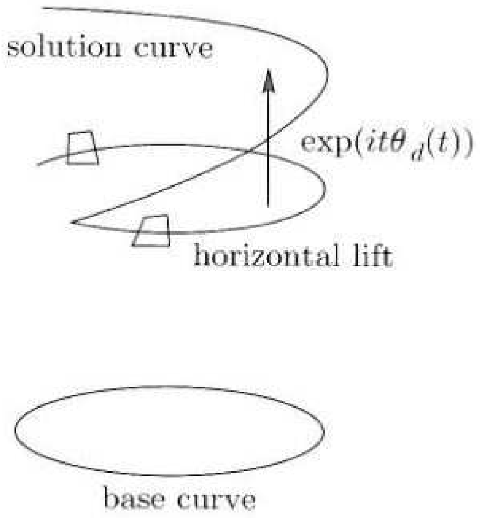
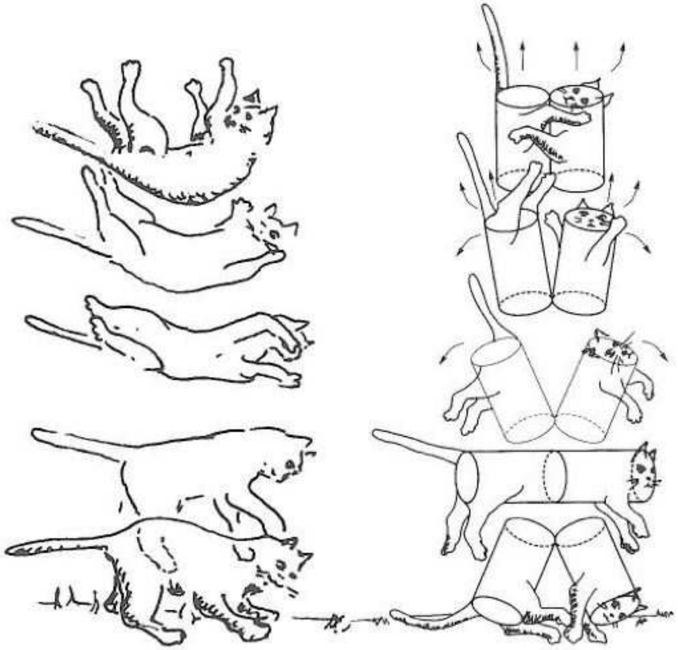
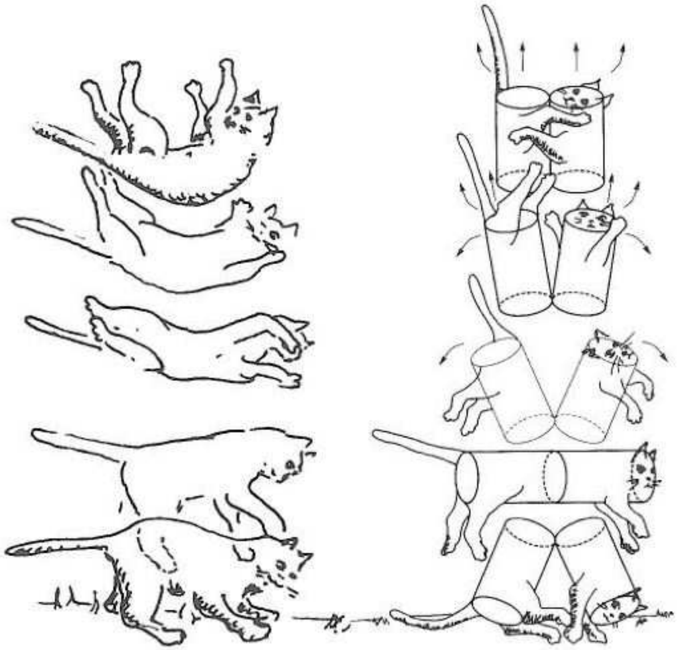


FIGURE 11.1. Dynamic versus horizontal lift.

# Falling cat



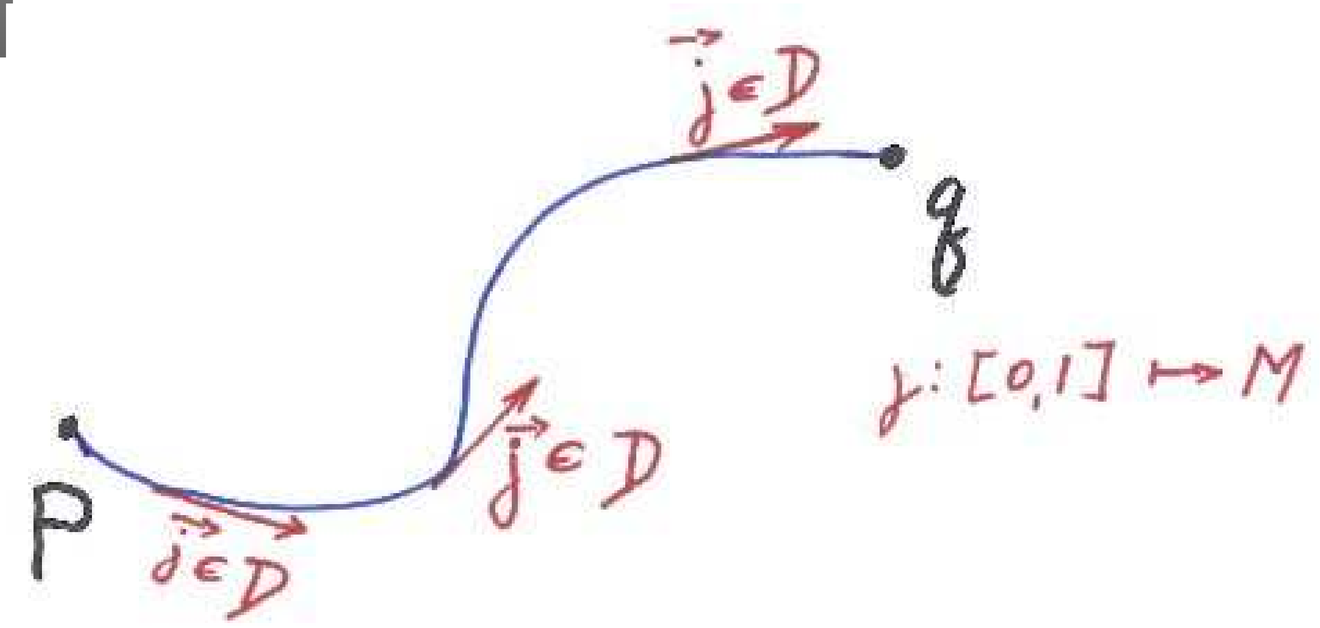
# Falling cat



$M$ is a space of "shape of the cat" together with "her orientation" and "location in the space". The group $G = SE(3)$ is the group of rigid motions that actually can be reduced to $SO(3)$. Thus $Q = M/G$ is a space of pure shapes. Since the initial shape is the same as the final, the problem to find the optimal way of falling is to find the shortest loop in the space of pure shapes $Q$.

# Connection

$(M^n, \mathcal{D}^k, \langle \cdot, \cdot \rangle_{\mathcal{D}^k})$ is a sub-Riemannian manifold



Can we join any points by a curve?

# *Bracket generating condition*

A distribution $\mathcal{D}^k = \mathrm{span}\{X_1, \dots, X_k\}$ is called bracket generating if $X_1, \dots, X_k$ together with all of its iterated Lie brackets $[X_i, X_j], [X_i, [X_j, X_m]], \dots$ span the tangent bundle $TM^n$.

If $\mathcal{D}^k$ is bracket generating and $M^n$ is connected, then any two points can be connected by a horizontal curve $\dot{\gamma} \in \mathcal{D}^k$.

Chow-Rashevskii theorem (1938-1939)

**1. W. L. Chow**: Uber Systeme von linearen partiellen Differentialgleichungen erster Ordnung, *Math. Ann.*, **117** (1939), 98-105.

**2. P. K. Rashevskii**: About connecting two points of complete nonholonomic space by admissible curve, *Uch. Zapiski ped. inst. Libknekhta*, **2** (1938), 83-94. in Russian

# Carnot-Carathéodory distance

Given bracket generating $\mathcal{D}^k$ and connected $M^n$ there is a

$$\gamma : [0,1] \to M : \quad \gamma(0) = x,\ \gamma(1) = y,\ \ \forall x, y \in M^n$$

$$\text{such that} \quad \dot\gamma(t) = \sum_{i=1}^{k} \alpha_i(t) X_i(\gamma(t))$$

The Carnot-Carathéodory distance is

$$d_{c-c}(x,y) = \inf\{ \int_0^1 \langle \dot\gamma(t), \dot\gamma(t) \rangle_{\mathcal{D}^k}^{1/2} \, dt :$$

$\gamma$ is horizontal and connects the points $x$ and $y\}$.

# Example. Heisenberg group

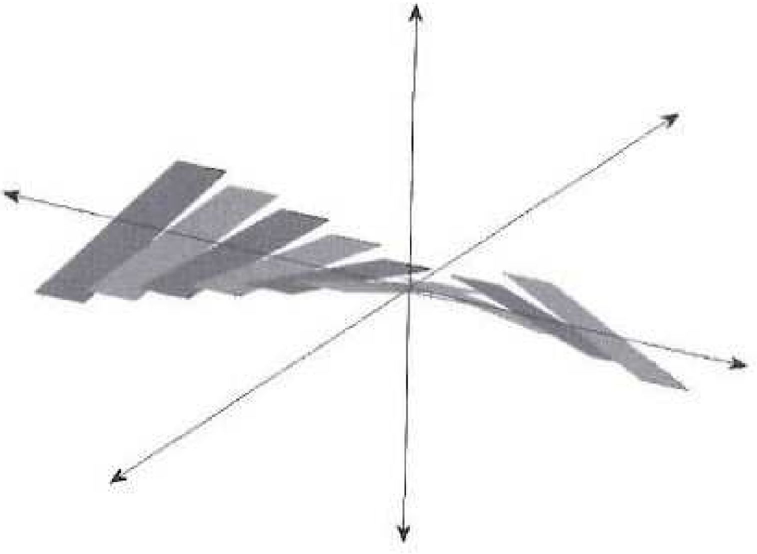
$$\mathbb{R}^3, \quad X = \partial_x - \tfrac{1}{2} y \, \partial_z, \quad Y = \partial_y + \tfrac{1}{2} x \, \partial_z,$$

$$Z = [X, Y] = XY - YX = \partial_z,$$

$$\operatorname{span}\{X, Y, [X, Y]\} = \mathbb{R}^3, \quad ds^2 = dx^2 + dy^2$$

$(\mathbb{R}^3, \mathbb{R}^2 = \operatorname{span}\{X, Y\}, ds^2)$ is the Heisenberg group

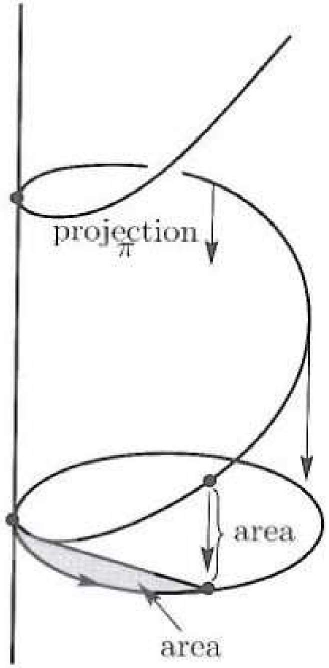
$$(\mathbb{R}^3, +) \quad \Rightarrow \quad (\mathbb{R}^3, \circ)$$



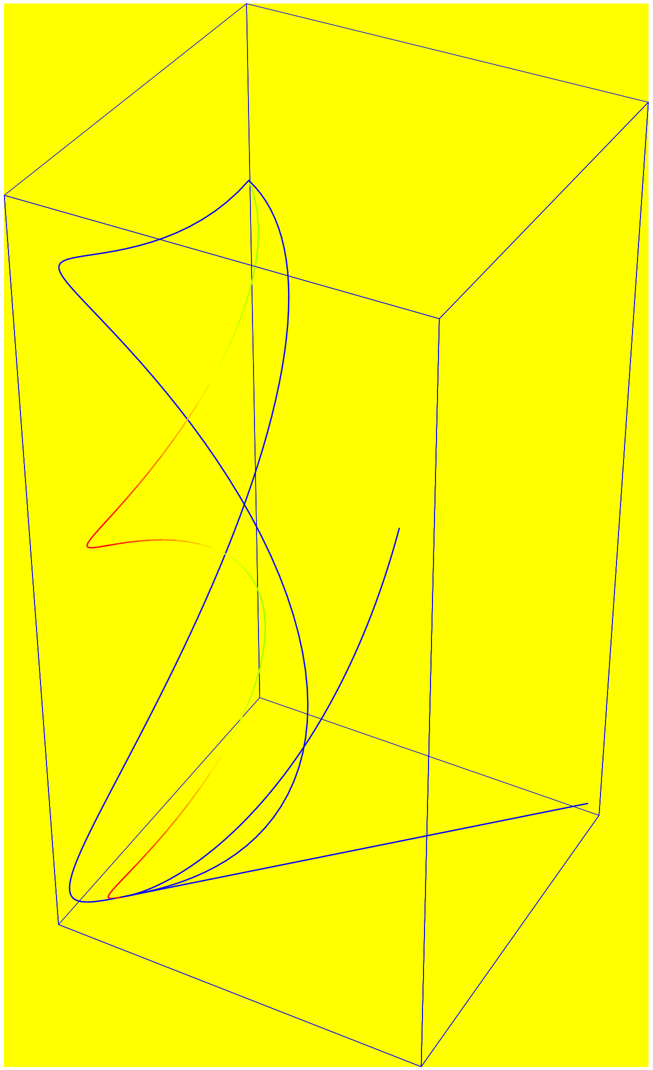
# Horizontal curve on Heisenberg group

$$\dot\gamma = \dot x\,\partial_x + \dot y\,\partial_y + \dot z\,\partial_z = \dot x X + \dot y Y + \Big(\dot z + \frac{1}{2}(y\dot x - x\dot y)\Big) Z$$

The horizontality condition $\dot z = \frac{1}{2}(x\dot y - y\dot x)$



# Geodesics



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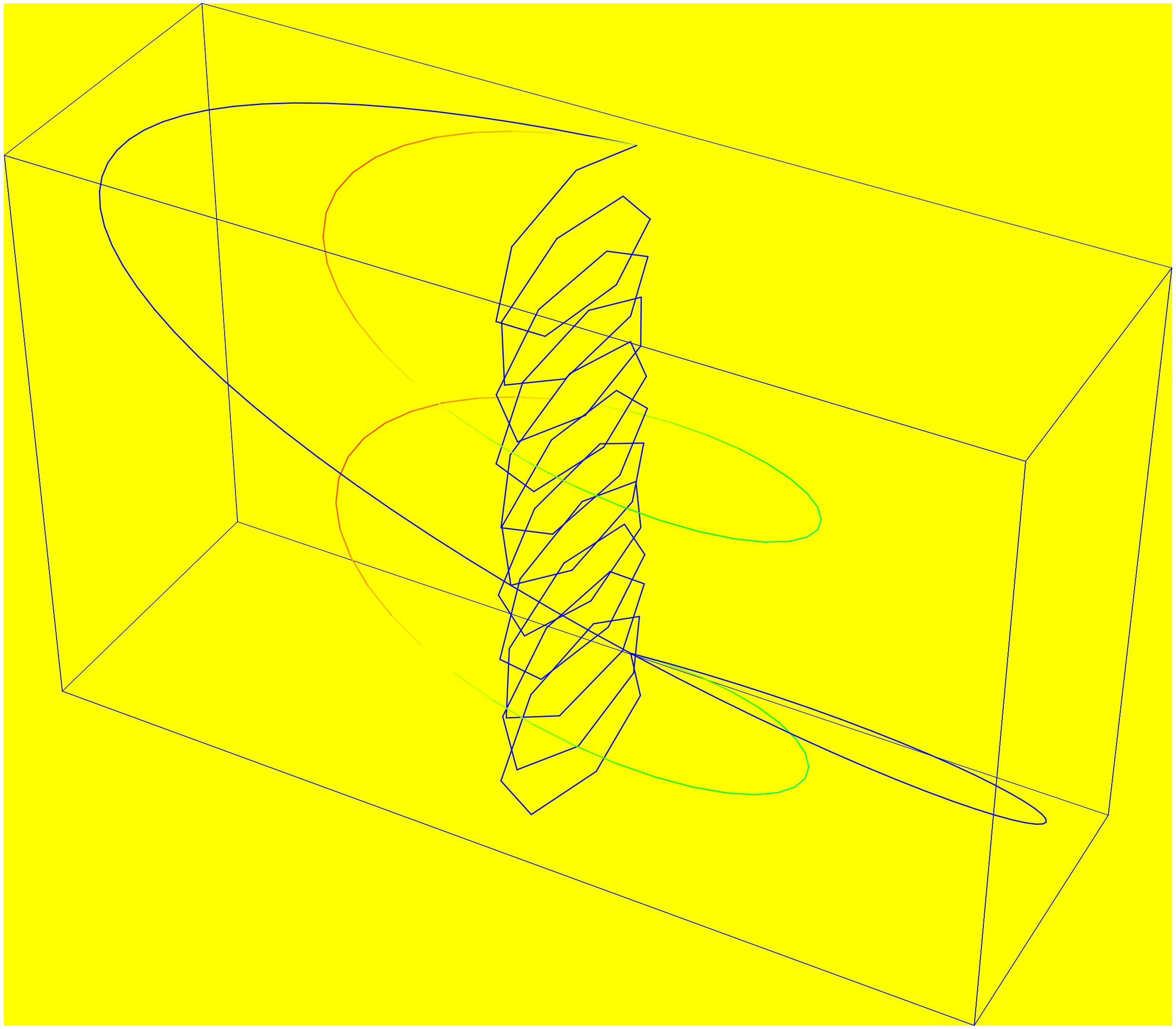

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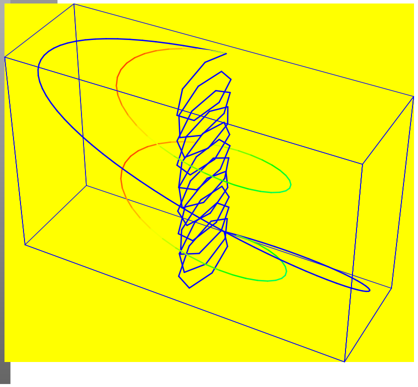


$$\omega = dz - \frac{1}{2}(x\,dy - y\,dx), \quad \Omega = d\omega = dx \wedge dy, \quad d\Omega = 0$$

$$\frac{d\overrightarrow{v}}{dt} = \overrightarrow{v} \times \Omega, \qquad \overrightarrow{v} = (\dot{x}, \dot{y}),$$

$$\Omega = 0dx \wedge dz + 0dz \wedge dy + 1dx \wedge dy$$

# Heisenberg ball

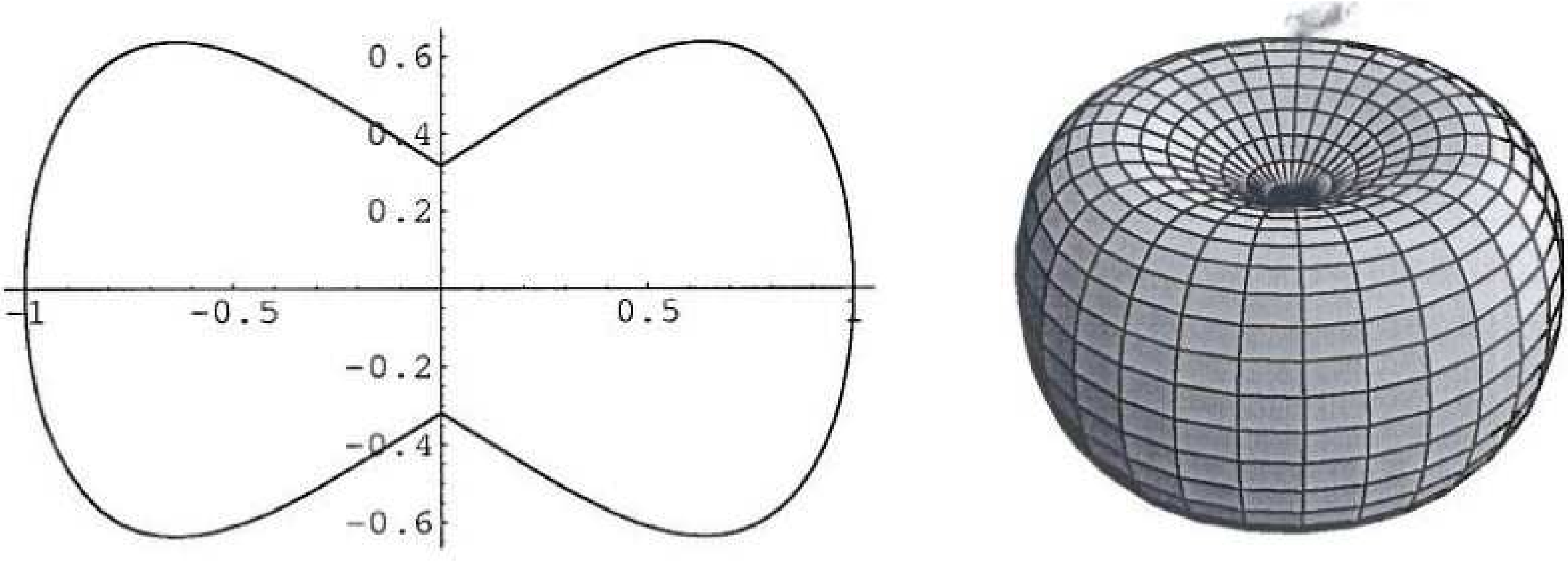


FIGURE 1. The unit ball in the Carnot-Carathéodory metric

## $SU(2)$ **group**

$$SU(2): \quad \begin{bmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{bmatrix}, \quad z_1, z_2 \in \mathbb{C}, \quad |z_1|^2 + |z_2|^2 = 1.$$

$(z_1, z_2)^{-1} = (\bar{z}_1, -z_2),$ $(1,0)$ is the unit

$S^3 = \{\overline{x} \in \mathbb{R}^4 : \; x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$ for

$$z_1 = x_1 + ix_2, \quad z_2 = x_3 + ix_4$$

$U(1, \mathbb{H})$ is the group of unit quaternions, $Sp(1)$ is the special symplectic group, $Spin(3)$ is the spin group on three generators. And they are double cover of the group $SO(3)$.

# Left invariant vector fields

$$\Big(L_q(\cdot)\Big)_* = \begin{bmatrix} x_1 & -x_2 & -x_3 & -x_4 \\ x_2 & x_1 & -x_4 & x_3 \\ x_3 & x_4 & x_1 & -x_2 \\ x_4 & -x_3 & x_2 & x_1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$N = x_1\partial_1 + x_2\partial_2 + x_3\partial_3 + x_4\partial_4$ is normal vector to $S^3$, $\langle N, N\rangle = 1$,

$Z = -x_2\partial_1 + x_1\partial_2 + x_4\partial_3 - x_3\partial_4$, $\langle Z, Z\rangle = 1$,

$X = -x_3\partial_1 - x_4\partial_2 + x_1\partial_3 + x_2\partial_4$, $\langle X, X\rangle = 1$,

$Y = -x_4\partial_1 + x_3\partial_2 - x_2\partial_3 + x_1\partial_4$, $\langle Y, Y\rangle = 1$,

$[Z, X] = 2Y$, $[Y, Z] = 2X$, $[X, Y] = 2Z$

# Horizontality condition

$$T_p(S^3) = \mathrm{span}\{X, Y, Z\}, \qquad \mathcal{D} = \mathrm{span}\{X, Y\}.$$

The geometry obtained by fixing other pair of vector fields is similar. Let $\gamma(s) = (x_1(s), x_2(s), x_3(s), x_4(s))$ be a curve on $S^3$. Then

$$\begin{aligned} \dot{\gamma} &= \dot{x}_1\partial_1 + \dot{x}_2\partial_2 + \dot{x}_3\partial_3 + \dot{x}_4\partial_4 \\ &= a(s)X(\gamma(s)) + b(s)Y(\gamma(s)) + c(s)Z(\gamma(s)). \end{aligned}$$

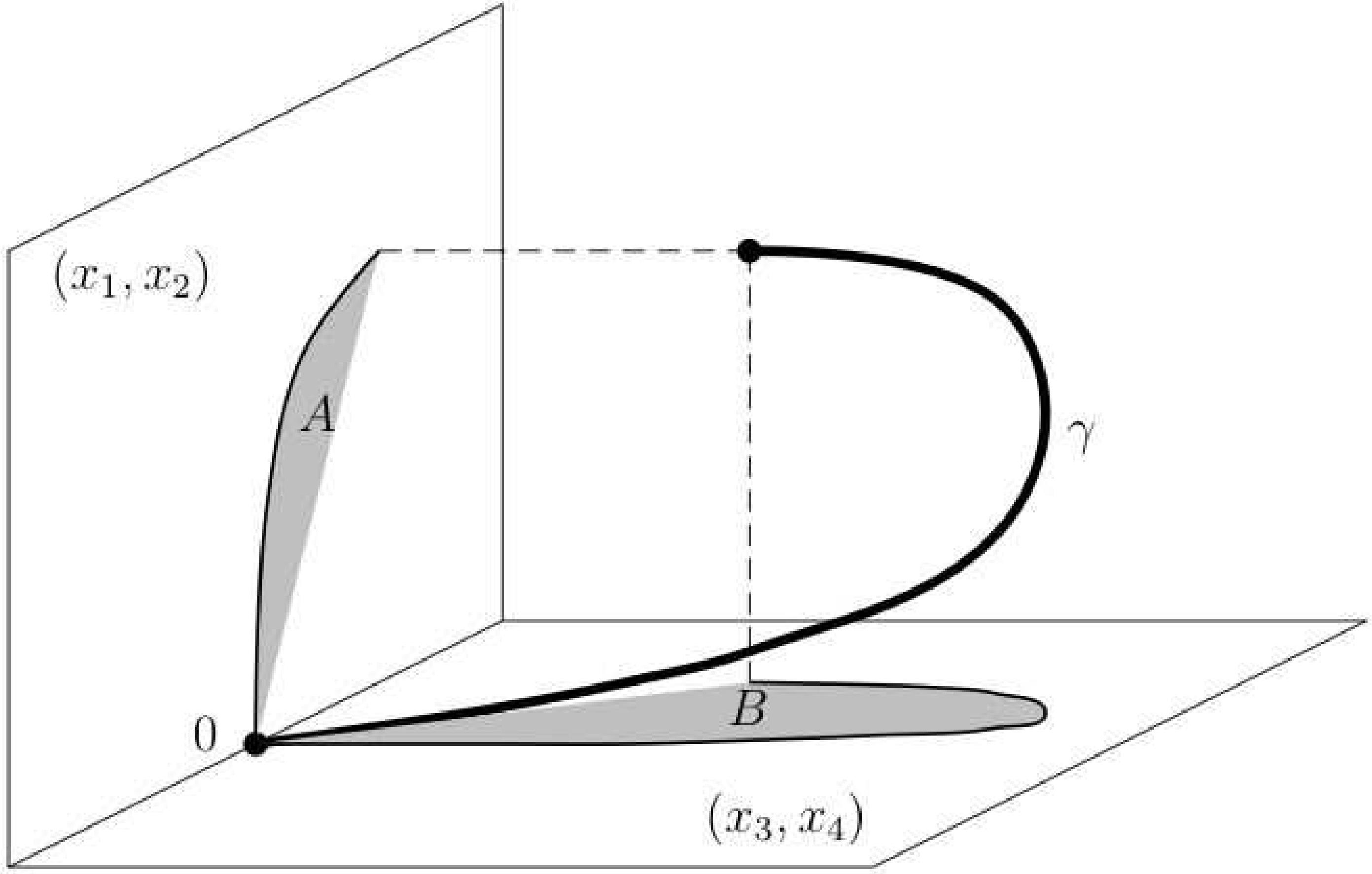
The curve $\gamma$ is horizontal iff

$$c = \langle \dot{\gamma}, Z \rangle = -x_2\dot{x}_1 + x_1\dot{x}_2 + x_4\dot{x}_3 - x_3\dot{x}_4 = 0.$$

The set of horizontal curves is not empty.

# Horizontality condition

$$\frac{1}{2}(x_1 dx_2 - x_2 dx_1) = \frac{1}{2}(x_3 dx_4 - x_4 dx_3)$$



# Hopf fibration

$h: \ S^3 \ \ \rightarrow \ \ S^2, \ h^{-1}(p) = S^1, \ \ p \in S^2$ is the principal $S^1$ bundle

$h(z_1, z_2) = (|z_1|^2 - |z_2|^2, 2z_1\bar{z}_2) \in S^2$. It is submersion of $S^3$ onto $S^2$ and bijection between $S^3/S^1$ and $S^2$. The action of $S^1$ on $S^3$ is defined by

$$e^{2\pi it} \cdot (z_1, z_2) = (e^{2\pi it} z_1, e^{2\pi it} z_2), \quad e^{2\pi it} \in S^1, \quad (z_1, z_2) \in S^3$$

$\phi(t) = e^{2\pi it} \cdot (\hat{z}_1, \hat{z}_2)$ is a fiber over $(\hat{z}_1, \hat{z}_2)$ that collapses to $h(\hat{z}_1, \hat{z}_2)$ under the Hopf map. The curve $\phi(t)$ is a great circle on $S^3$.

# Ehresmann connection

$$d_{\phi(t)}h(\dot{\phi}(t)) = 0$$

$$\ker(dh) = \mathrm{span}\{\dot{\phi}(t)\} = \mathrm{span}\{2\pi Z\} \subset TS^3$$

The orthogonal complement to $\ker(dh)$ is $\mathcal{D} = \mathrm{span}\{X, Y\}$:

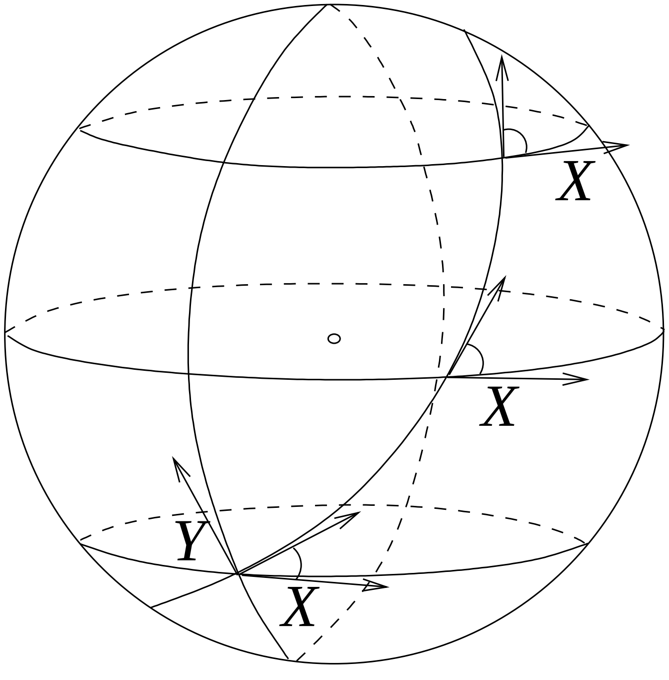
$$\ker(dh) \oplus \mathcal{D} = TS^3$$

The distribution $\mathcal{D}$ constructed in this way is called the Ehresmann connection. The metric on $\mathcal{D}$ coincides with the pull back of the metric on $S^2$ by the Hopf map.

# Geodesics

$$\int (a(s)^2 + b(s)^2)\,ds + \lambda(s)c(s)$$

The geodesics are characterized by the following. The angle $\angle(\dot\gamma, X(c(s)))$ increases linearly.



# Hamiltonian system

$$H = \frac{1}{2}(X^2 + Y^2) = \frac{1}{2}\Big((I_1 x, \xi)^2 + (I_2 x, \xi)^2\Big).$$

$$\dot{x} = \frac{\partial H}{\partial \xi}, \qquad \dot{\xi} = -\frac{\partial H}{\partial x}$$

$$\ddot{x}_k = -x_k, \quad a^2 + b^2 = \cos^2\psi + \sin^2\psi = 1, \quad k = 1, 2, 3, 4.$$

$$x_1 = \cos s, \quad x_2 = 0, \quad x_3 = \cos\psi \sin s, \quad x_4 = \sin\psi \sin s$$

horizontal "plane" at $x(0) = (1, 0, 0, 0)$.

$x_1 = \cos s,\ x_2 = \sin s,\ x_3 = 0,\ x_4 = 0$ is the vertical line.

# Equations for geodesics

Introduce the complex coordinates

$$z_1 = x_1 + ix_2,\ z_2 = x_3 + ix_4,\ \varphi = \xi_1 + i\xi_2,\ \psi = \xi_3 + i\xi_4.$$

$H = \frac{1}{2}|\bar{z}_2\varphi - z_1\bar{\psi}|^2$

$$\begin{aligned} z_1(s) &= \Big(\cos(s|\dot{z}_2(0)|\sqrt{1+k^2}) \\ &+ i\frac{k}{|\dot{z}_2(0)|\sqrt{1+k^2}}\sin(s|\dot{z}_2(0)|\sqrt{1+k^2})\Big)e^{-i|\dot{z}_2(0)|ks}, \end{aligned}$$

$$z_2(s) = \frac{\dot{z}_2(0)}{|\dot{z}_2(0)|\sqrt{1+k^2}}\sin(s|\dot{z}_2(0)|\sqrt{1+k^2})e^{i|\dot{z}_2(0)|ks}.$$

# Clifford torus

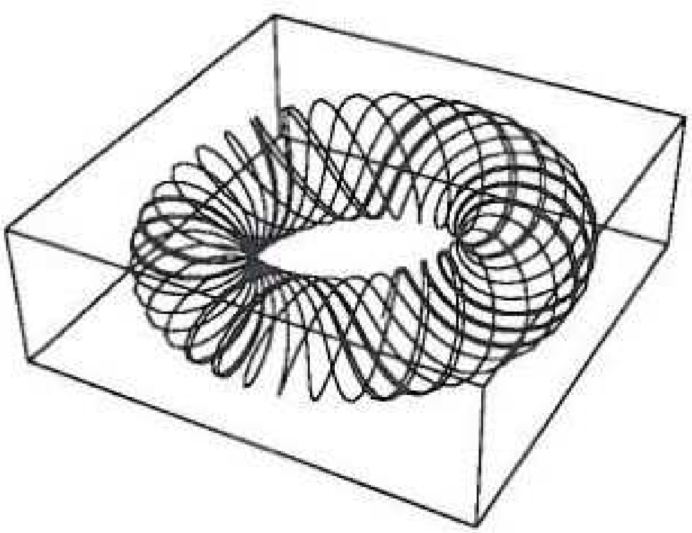


**Fig. 1** Stereographic projection from $\mathbb{S}^3$—{north pole} to $\mathbb{R}^3$ of a sub-Riemannian geodesic which is dense inside a Clifford torus

Clifford torus $= \{(z_1, z_2) \in S^3 : \; |z_1|^2 = \rho^2\}$. If the curvature is rational, then the geodesics are diffeomorphic to a circle otherwise it is diffeomorphic to a straight line and it is dense subset inside the Clifford torus.

# Hypoellipticity

Laplacian $\Rightarrow$ sub-Laplacian

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \qquad \Rightarrow \qquad \Delta_h = \sum_{i=1}^{k} X_i^2, \ k < n$$

# *Hypoellipticity*

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THEOREM. If the vector fields $\{X_1, \dots, X_k\}$ are real and if they satisfy the bracket generating condition then the operator $\Delta_h = \sum_{i=1}^{k} X_i^2$ is hypoelliptic:

$$\Delta_h u = f : \quad f \in C^\infty \quad \Rightarrow \quad u \in C^\infty.$$

**Hörmander L.** Hypoelliptic second order differential equations. *Acta Math*, **119** (1967), 147–171.

# Sub-ellipticity

The Laplace operator is elliptic, it gains 2 derivatives in Sobolev norm:

$$\|u\|_{W_2^2} \leq C\Big(|(\Delta u, u)| + \|u\|_{L_2}\Big) \quad \forall u \in C_0^\infty.$$

The sub-elliptic means that you can not have the estimates in $W_2^2$, but only in $W_2^s$ with $s < 2$.

# Semigroup of sub-Laplacian

We are interested in finding of a closed form of the kernel $P_t(x, x_0)$ of the semigroup

$$\exp(-t\Delta_h), \qquad \Delta_h = X^2 + Y^2$$

on the group $SU(2)$.

Spectral method, representation group theory, Laguerre calculus, path integral method, Hamilton-Jacobi method.

# A bit of known theory

HAMILTONIAN'S PRINCIPLE OF LEAST ACTION Motion of mechanical system coincides with extremals of the functional $\int_\gamma L(q,\dot q,t)$. Extremals can be found from

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot q}\Big)-\frac{\partial L}{\partial q}=0 \qquad (E-L)$$

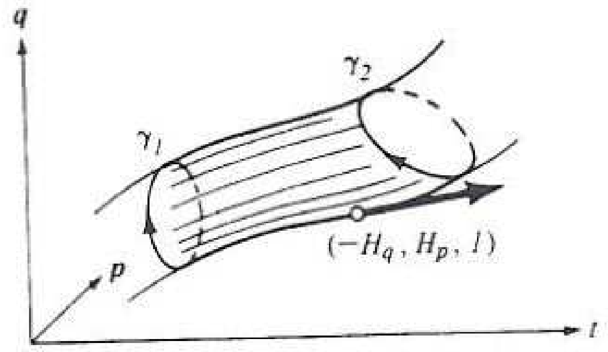
The system $(E-L)$ is equivalent to $2n$ equations

$$\begin{aligned}\dot p &= -\frac{\partial H}{\partial q}\\ \dot q &= \frac{\partial H}{\partial p},\end{aligned}$$

$H(p,q,t)=p\dot q-L(q,\dot q,t)$, $\qquad p=\frac{\partial L}{\partial \dot q}$.

# A bit of known theory

$dS = pdq - Hdt$ is an integral invariant of Poincaré-Cartan



Action $S$ satisfies the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + H(q, t, \frac{\partial S}{\partial q}) = 0 \qquad (H - J)$$

# **Relation between $(H)$ and $(H-J)$**

In order to solve the $(H)$ system we look for the canonical transformation

$$(p,q) \quad \to \quad (P,Q)$$

The generating function is a solution of the $(H-J)$ type equation.

Thus given a solution of $(H-J)$ equation we get a canonical transformation and can solve the $(H)$ system in quadratures.

# Relation between $(H)$ and $(H-J)$

To solve the Cauchy problem

$$\frac{\partial S}{\partial t} + H(q, t, \frac{\partial S}{\partial q}) = 0, \quad S(q, t_0) = S_0(q)$$

we look for the solution of $\dot{p} = -H_q, \quad \dot{q} = H_p$ with initial conditions

$$q(t_0) = q_0, \qquad p(t_0) = \frac{\partial S_0}{\partial q}\Big|_{q_0}$$

# Relation between $(H)$ and $(H-J)$

The solution of $(H)$ system is a an extremal for principle $\delta\int L\,dt=0$, where the Lagrangian and Hamiltonian is related by the Legandre transformation.

It is called characteristic for $(H-J)$. Then

$$S(q)=S_0(q_0)+\int_{q_0,t_0}^{q} L(q,\dot{q},t)\,dt$$

with the integration along the extremal.

## *Heat kernel en $\mathbb{R}^n$*

Let $\Delta - \frac{\partial}{\partial t}$ be a heat operator, then

$$P_t(x, x_0) = \frac{1}{(2\pi t)^{\frac{n}{2}}} e^{-\frac{|x-x_0|^2}{2t}}.$$

If $f = \frac{1}{2}|x - x_0|^2$, then $\frac{f}{t}$ satisfies the Hamilton-Jacobi equation

$$\frac{\partial}{\partial t}\Big(\frac{f}{t}\Big) = -\frac{1}{2}\sum_{j=1}^{n}\Big(\frac{\partial}{\partial x_j}\Big(\frac{f}{t}\Big)\Big)^2 = H\Big(\nabla\big(\frac{f}{t}\big)\Big),$$

where $H$ is associated with $\Delta$. The function $S = \frac{f}{t}$ is the classical action related to the Hamiltonian $H$.

# General vector fields

Let $X_j$, $j = 1, \dots, n$ be smooth linearly independent vector fields in $\mathbb{R}^n$.

$$P_t(x, x_0) = \frac{1}{(2\pi t)^{\frac{n}{2}}} e^{-\frac{|x-x_0|^2}{2t}} (v_0 + v_1 t + v_2 t^2 + \dots),$$

where the function $\frac{|x-x_0|^2}{2t}$ satisfies the Hamilton-Jacobi equation with respect to the vector fields $X_j$.

# Heisenberg group

$$X_1 = \partial_{x_1} - \tfrac{1}{2} x_2 \, \partial_z, \qquad X_2 = \partial_{x_2} + \tfrac{1}{2} x_1 \, \partial_z,$$

$Z = [X_1, X_2] = 2 \, \partial_z$

Consider sub-Laplacian $\Delta_X = X_1^2 + X_2^2$. It is not elliptic, but still hypoelliptic and sub-eliptic. The heat kernel has the form

$$P_t(x, z) = \frac{1}{(2\pi t)^2} \int_{-\infty}^{\infty} e^{-\frac{f(x,z,\tau)}{t}} V(\tau) \, d\tau$$

$$\tau \frac{\partial f}{\partial \tau} + H(\nabla_X f) = f$$

$V(\tau)$ is a volume element.

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# Hyperspherical coordinates

$$\begin{aligned} x_1 + ix_2 &= e^{i\xi_1}\cos\eta, \\ x_3 + ix_4 &= e^{i\xi_2}\sin\eta, \qquad \eta \in [0, \pi/2], \quad \xi_1, \xi_2 \in [0, 2\pi), \end{aligned}$$

The horizontality condition is

$$\dot\xi_1 \cos^2\eta - \dot\xi_2 \sin^2\eta = 0.$$

The horizontal 2-sphere is obtained from the parametrization, if we set $\xi_1 = 0$, $\xi_2 = \psi$, $\eta = s$.

The vertical line is obtained from the parametrization setting $\eta = 0$, $\xi_1 = s$.

# Vector fields and Hamiltonian

$$X=\sin(\xi_1-\xi_2)\tan\eta\partial_{\xi_1}+\sin(\xi_1-\xi_2)\cot\eta\partial_{\xi_2}+2\cos(\xi_1-\xi_2)\partial_\eta,$$

$$Y=\cos(\xi_1-\xi_2)\tan\eta\partial_{\xi_1}+\cos(\xi_1-\xi_2)\cot\eta\partial_{\xi_2}-2\sin(\xi_1-\xi_2)\partial_\eta.$$

$$Z=\partial_{\xi_1}-\partial_{\xi_2},$$

$$\frac{1}{2}(X^2+Y^2)\quad\Rrightarrow\quad H=\frac{1}{2}\big((\tan\eta\psi_1+\cot\eta\psi_2)^2+4\theta^2\big)$$

$$\psi_i=\partial_{\xi_i},\qquad\theta=\partial_\eta$$

$$ChVar_{(\xi_1,\xi_2,\eta)}:=\{\psi_1=\tau\cot\eta,\ \psi_2=-\tau\tan\eta,\ \theta=0\}$$

# Hamiltonian system

$$\begin{aligned}
\dot{\xi}_1 &= \frac{\partial H}{\partial \psi_1} = \psi_1 \tan^2 \eta + \psi_2 \\
\dot{\xi}_2 &= \frac{\partial H}{\partial \psi_2} = \psi_2 \cot^2 \eta + \psi_1 \\
\dot{\psi}_1 &= -\frac{\partial H}{\partial \xi_1} = 0 \\
\dot{\psi}_2 &= -\frac{\partial H}{\partial \xi_2} = 0
\end{aligned}$$

$$\begin{aligned}
\dot{\eta} &= \frac{\partial H}{\partial \theta} = 4\theta \\
\dot{\theta} &= -\frac{\partial H}{\partial \eta} = -\psi_1^2 \frac{\tan \eta}{\cos^2 \eta} + \psi_2^2 \frac{\cot \eta}{\sin^2 \eta}
\end{aligned}$$

$$\eta(0) = \eta_0,\ \eta(s) = \eta, \qquad \xi_1(s) = \xi_1,\ \xi_2(s) = \xi_2,$$
$$\psi_1(0) = \psi_1,\ \psi_2(0) = \psi_2.$$

# Action function

$$f(\xi_1, \xi_2, \eta_0, \eta, \psi_1, \psi_2) = \psi_1 \xi_1 + \psi_2 \xi_2 + \int_0^1 (\theta \dot{\eta}(s) - H) ds.$$

It does satisfy the $(H - J)$ type equation

$$\psi_1 \frac{\partial f}{\partial \psi_1} + \psi_2 \frac{\partial f}{\partial \psi_2} + H(\xi_1, \xi_2, \eta, \nabla f) = f.$$

and play the role of the square of the distance from a fixed position $((\xi_1)_0, (\xi_2))_0, \eta_0)$ at the critical points $\psi_1^*, \psi_2^*$

# Action function

$$f = \psi_1\xi_1 + \psi_2\xi_2 + \frac{1}{2}A + \frac{1}{2}(\psi_1 - \psi_2)^2$$

$$-\frac{1}{2}\psi_1 \arctan\frac{\sqrt{A}}{\psi_1}\left\{\left[1 - \frac{1}{2}\left(1 + \frac{\psi_2^2 - \psi_1^2}{A}\right)\right]\tan\left(2\sqrt{A} + \frac{D_1}{2}\right) \mp$$

$$+\frac{1}{2}\psi_1 \arctan\frac{\sqrt{A}}{\psi_1}\left\{\left[1 - \frac{1}{2}\left(1 + \frac{\psi_2^2 - \psi_1^2}{A}\right)\right]\tan\frac{D_1}{2} \mp D_0\right\}$$

$$-\frac{1}{2}\psi_2 \arctan\frac{\sqrt{A}}{\psi_2}\left\{\frac{1}{2}\left(1 + \frac{\psi_2^2 - \psi_1^2}{A}\right)\tan\left(2\sqrt{A} + \frac{D_1}{2}\right) \pm D_0\right\}$$

$$+\frac{1}{2}\psi_2 \arctan\frac{\sqrt{A}}{\psi_2}\left\{\frac{1}{2}\left(1 + \frac{\psi_2^2 - \psi_1^2}{A}\right)\tan\frac{D_1}{2} \pm D_0\right\}.$$

# Heat kernel on the 3-D sphere

$$P_t\big((0,0,\frac{\pi}{4}),(\xi_1,\xi_2,\eta)\big) = \frac{1}{(2\pi t)^\alpha} \int_{ChVar_{(0,0,\frac{\pi}{4})}} e^{-\frac{f(\xi_1,\xi_2,\eta,\tau)}{t}} V(\eta,\tau)\, d\tau$$

$$f(\tau,\xi_1,\xi_2,\eta) = \tau(\xi_1 - \xi_2) + \frac{A}{2} + 2\tau^2 - \frac{\tau}{2} \arctan\Big(\frac{2\tau}{\sqrt{A}} \tan 4\sqrt{A}\Big),$$

$$\cos 2\eta = -\sqrt{1 - \frac{4\tau^2}{A}}(\sin 4\sqrt{A})$$

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# The end

Thank you for your attention