Two Rigidity Theorems for Holomorphic Generators of Continuous Semigroups

Main notations:

$\Delta := \{z \in \mathbf{C} : |z| < 1\}$ $\qquad$ $\mathbf{B} := \{x \in H : \|x\| < 1\}$

$\overline{\Delta} := \{z \in \mathbf{C} : |z| \le 1\}$ $\qquad$ $\overline{\mathbf{B}} := \{x \in H : \|x\| \le 1\}$

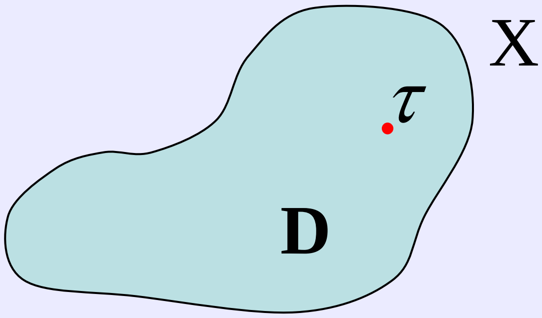
$\partial\Delta = \{z \in \mathbf{C} : |z| = 1\}$ $\qquad$ $\partial\mathbf{B} := \{x \in H : \|x\| = 1\}$

$\langle x, y \rangle$ - the inner product of $x, y \in \mathbf{B}$

$\mathrm{Hol}(D, E)$ - the set of all holomorphic mappings on a domain $D \subset H$ which map D into a set $E \subset H$

$\mathrm{Hol}(D) := \mathrm{Hol}(D, D)$ - the set of all holomorphic self-mappings of D

The rigidity problem:



Given: a Banach space X

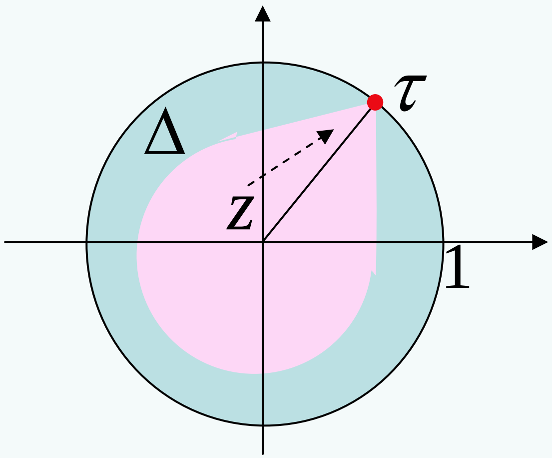
a domain $D \subset X$

a mapping $G : D \mapsto X$

The question:

$$\left.\begin{array}{l} F : D \mapsto X \\ F(\tau) = G(\tau), \quad \tau \in \overline{D} \\ F^{(k)}(\tau) = G^{(k)}(\tau) \end{array}\right\} \overset{?}{\Rightarrow} \quad F = G \quad \text{on } D$$

Angular limits and K-limits:



The angular approach region at τ .

$$\Gamma_\sigma(\tau) := \{z \in \Delta : \ |z-\tau| < \sigma(1-|z|)\}, \quad \sigma > 1$$

Angular limit $\quad g : \Delta \to \mathbf{C}$

$$\angle \lim_{z \to \tau} g(z) =: g(\tau)$$

$$\angle \lim_{z \to \tau} g'(z) =: g'(\tau)$$

The Korányi approach region at τ .

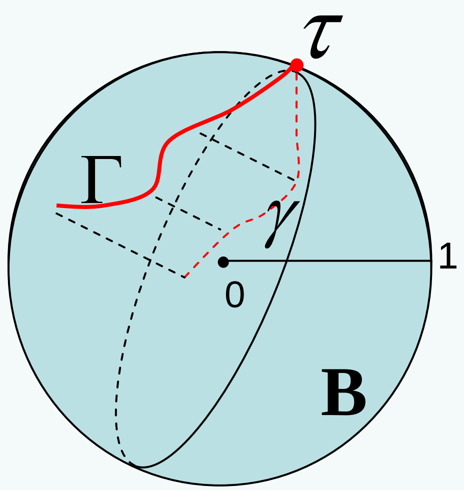
$$D_\alpha(\tau) := \left\{ x \in \mathbf{B} : \left|1 - \langle x, \tau \rangle\right| < \frac{\alpha}{2}\left(1 - \|x\|^2\right) \right\}$$

K-limit $\quad f : \mathbf{B} \to H$

$$K - \lim_{x \to \tau} f(x) =: f(\tau)$$

$$K - \lim_{x \to \tau} f'(x) =: f'(\tau)$$

Restricted K-limit



Restricted curve Γ ending at τ :

$$\Gamma : [0,1) \mapsto \mathbf{B}$$

$$\gamma(t) = \langle \Gamma(t), \tau \rangle \tau$$

1. $\displaystyle \lim_{t \to 1^-} \frac{\|\Gamma(t) - \gamma(t)\|^2}{1 - \|\gamma(t)\|^2} = 0$

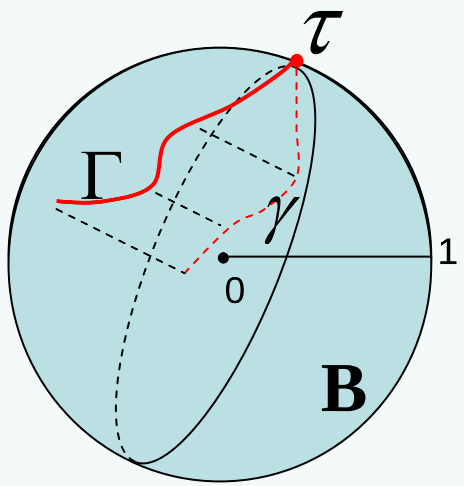
2. $\displaystyle \frac{\|\gamma(t) - \tau\|}{1 - \|\gamma(t)\|} \le A < \infty \quad (0 \le t < 1)$

Restricted K-limit L of $f : \mathbf{B} \mapsto H$ at τ :

If for each restricted curve Γ ending at τ , $\displaystyle \lim_{t \to 1^-} f\big(\Gamma(t)\big) = L$.

Remark. f may have a restricted K-limit at $\tau \in \partial \mathbf{B}$ without having a K-limit at τ .

Weak restricted K-limit



We say that the **weak restricted K-limit** of $f:\mathbf{B}\mapsto H$ at $\tau\in\partial\mathbf{B}$ equals $M\in H$ if for each $u\in H$,

$$\langle f(x),u\rangle \to \langle M,u\rangle$$

as x tends to τ along every restricted curve ending at τ.

Semigroups: $\{F_t\}_{t\geq 0} \subset \mathrm{Hol}(D), \quad D \subset H$

(i) $F_t(F_s(x)) = F_{t+s}(x), \quad \forall s,t \geq 0, \quad \forall x \in D$

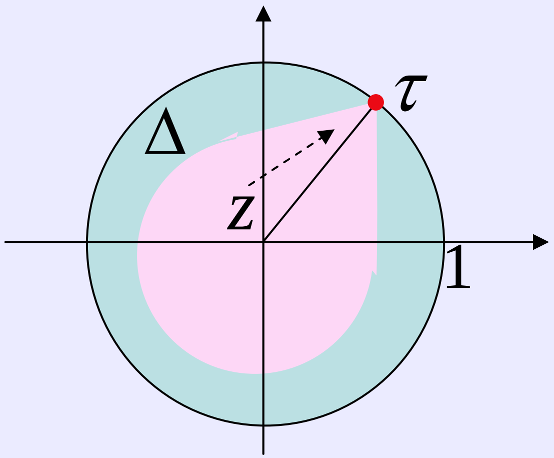
(ii) $\lim_{t\to 0^+} F_t(x) = x, \quad \forall x \in D$

Generators:

$$f(x) := \lim_{t\to 0^+} \frac{1}{t}\left(x - F_t(x)\right), \quad f : D \mapsto H$$

Proposition 1. *Let* $f \in Hol(\Delta, \mathbf{C})$ *be the generator of a one-parameter continuous semigroup on* Δ. *Suppose that for some* $\tau \in \partial\Delta$

$$\angle\lim_{z \to \tau} \frac{f(z)}{|z-\tau|^3} = 0.$$

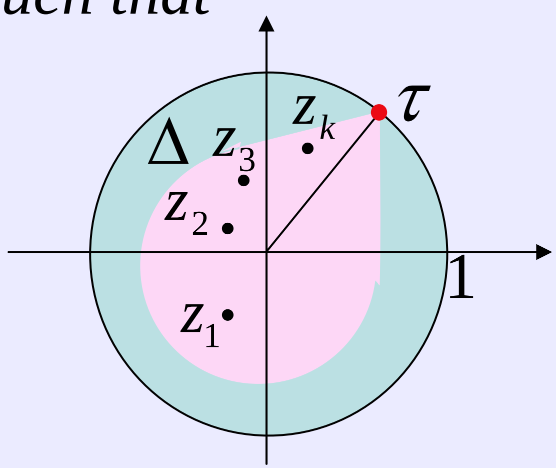


Then $f = 0$ *on* Δ.

$$f(\tau) = f'(\tau) = f''(\tau) = f'''(\tau) = 0$$

Theorem 1. *Let* $f \in Hol(\Delta, \mathbf{C})$ *be the generator of a one-parameter continuous semigroup on* Δ. *If for some* $\tau \in \partial\Delta$, *there is a sequence* $\{z_k\}_{k=1}^{\infty} \subset \Delta$ *converging nontangentially to* τ *such that*

$$\lim_{k \to \infty} \frac{f(z_k)}{|z_k - \tau|^3} = 0,$$



then $f(z) \equiv 0$ *on* Δ.

Proposition 2. *Let* $f \in Hol(\mathbf{B}, H)$ *be the generator of a one-parameter continuous semigroup on* $\mathbf{B}$. *If for some* $\tau \in \partial\mathbf{B}$,

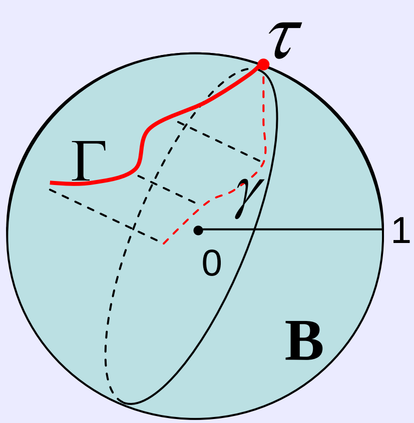
$$K - \lim_{x \to \tau} \frac{f(x)}{\|x - \tau\|^3} = 0,$$

then $f(x) \equiv 0$ *on* $\mathbf{B}$.

Theorem 2. *Let* $f \in Hol(\mathbf{B}, H)$ *be the generator of a one-parameter continuous semigroup on* $\mathbf{B}$. *If for some* $\tau \in \partial\mathbf{B}$, *the weak restricted K-limit of*

$$\frac{f(x)}{\|x - \tau\|^3}$$

equals 0, then $f(x) \equiv 0$ *on* $\mathbf{B}$.



Lemma 1. *Let* $p \in \mathbf{Hol}(\Delta, \overline{\Pi}^{+})$, *where* $\Pi^{+} := \{z \in \mathbf{C} : \operatorname{Re} z > 0\}$. *Suppose that for some* $\tau \in \partial\Delta$, *there is a sequence* $\{z_k\}_{k=1}^{\infty} \subset \Delta$ *converging nontangentially to* τ *such that the limit*

$$\alpha := \lim_{k\to\infty} \frac{p(z_k)}{1-\bar{\tau}z_k}$$

exists finitely. Then $\alpha \geq 0$. *Moreover, if* $\alpha = 0$, *then* $p(z) \equiv 0$ *on* Δ.

Lemma 2. *Let* $f \in \mathrm{Hol}(\Delta, \mathbf{C})$ *be a generator on* Δ. *Suppose there are a point* $\tau \in \partial\Delta$ *and a sequence* $\{z_k\}_{k=1}^{\infty} \subset \Delta$ *converging nontangentially to* τ *such that the limit*

$$\beta := \lim_{k\to\infty} \frac{f(z_k)}{z_k - \tau}$$

exists and is nonnegative. If f *does not vanish identically on* Δ, *then* τ *is the Denjoy-Wolff point of the semigroup generated by* f.

Proof. *Let* $\varsigma \in \overline{\Delta}$ *be the Denjoy-Wolff point of the semigroup. Then*

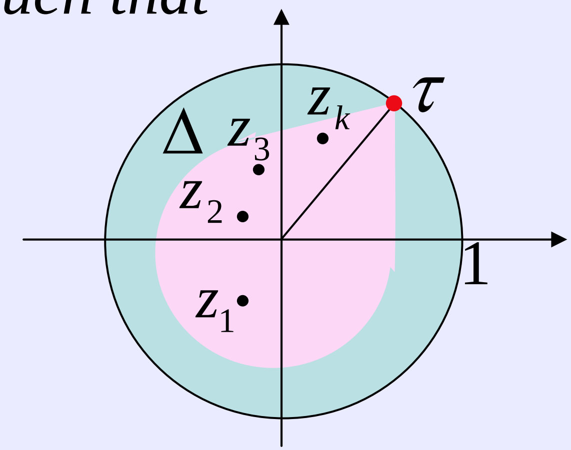
$$f(z) = (z-\varsigma)(1-\bar{\varsigma}z)\,p(z), \quad z \in \Delta, \quad \operatorname{Re} p(z) \geq 0.$$

$$\beta = \lim_{k\to\infty} \frac{f(z_k)}{z_k-\tau} = \lim_{k\to\infty} \frac{(z_k-\varsigma)(1-\bar{\varsigma}z_k)\,p(z_k)}{z_k-\tau} = \left|1-\bar{\varsigma}\tau\right|^2 \lim_{k\to\infty} \frac{p(z_k)}{\bar{\tau}z_k-1} \underset{Lemma 1}{\leq} 0$$

$$\Rightarrow \quad \beta = 0 \quad \Rightarrow \quad \tau = \varsigma \quad \text{or} \quad \lim_{k\to\infty} \frac{p(z_k)}{\bar{\tau}z_k-1} = 0$$

If $\displaystyle\lim_{k\to\infty} \frac{p(z_k)}{\bar{\tau}z_k-1} = 0 \underset{Lemma\ 1}{\Rightarrow} p \equiv 0 \Rightarrow f \equiv 0.$ *So* $\tau = \varsigma$.

Theorem 1. *Let* $f \in Hol(\Delta, \mathbf{C})$ *be the generator of a one-parameter continuous semigroup on* Δ*. If for some* $\tau \in \partial\Delta$*, there is a sequence* $\{z_k\}_{k=1}^{\infty} \subset \Delta$ *converging nontangentially to* τ *such that*



$$\lim_{k\to\infty} \frac{f(z_k)}{|z_k - \tau|^3} = 0,$$

then $f(z) \equiv 0$ *on* Δ.

Proof.

$$\lim_{k\to\infty} \frac{f(z_k)}{z_k - \tau} = 0 \underset{Lemma\,2}{\Rightarrow} \tau \text{ is the Denjoy-Wolff point of the semigroup}$$

$$\Rightarrow \quad f(z) = (z-\tau)(1-\bar{\tau}z)p(z), \quad \operatorname{Re} p(z) \geq 0, \quad z \in \Delta$$

$$\Rightarrow \quad \lim_{k\to\infty} \frac{f(z_k)}{(z_k-\tau)^3} = \lim_{k\to\infty} \frac{(z_k-\tau)(1-\bar{\tau}z_k)p(z_k)}{(z_k-\tau)^3} = -\bar{\tau} \lim_{k\to\infty} \frac{p(z_k)}{z_k-\tau}$$

$$\Rightarrow \quad \lim_{k\to\infty} \frac{p(z_k)}{z_k-\tau} = 0 \underset{Lemma\ 1}{\Rightarrow} p(z) \equiv 0 \Rightarrow f(z) \equiv 0.$$

$\forall \quad F \in \mathrm{Hol}(\Delta), \quad f = I - F$ is a holomorphic generator

$$F(z_k) = z_k + o\left(\|z_k - \tau\|^3\right) \text{ as } k \to \infty \quad \Rightarrow \quad F(z) \equiv z$$

Example. $F(w_1, w_2) = (w_1, 0)$

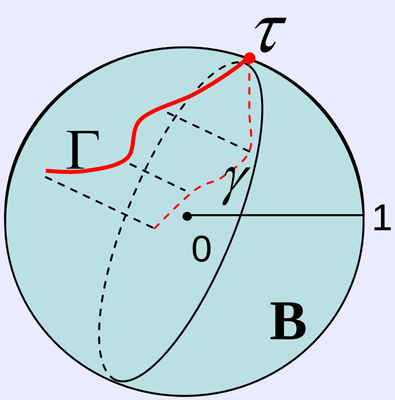
$$\tau = (1,0)$$

$$\{z_k\}_{k=1}^{\infty} = \{(1 - 1/k, 0)\}_{k=1}^{\infty}$$

$$F(z_k) = z_k \quad \forall k$$

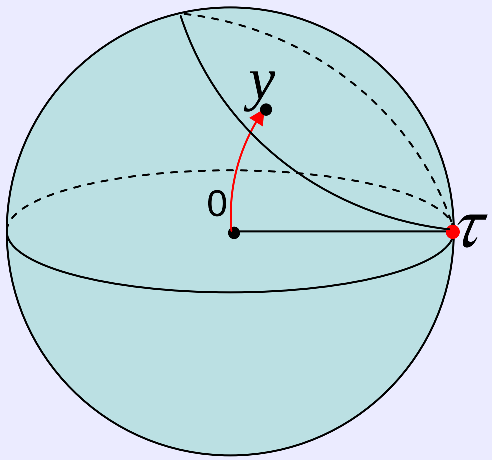
Theorem 2. *Let* $f \in Hol(\mathbf{B}, H)$ *be the generator of a one-parameter continuous semigroup on* $\mathbf{B}$. *If for some* $\tau \in \partial\mathbf{B}$, *the weak restricted K-limit of*

$$\frac{f(x)}{\|x-\tau\|^3}$$



equals 0, then $f(x) \equiv 0$ *on* $\mathbf{B}$.

Proof. $m \in \mathrm{Aut}(\mathbf{B}), \quad m(\tau) = \tau, \quad m(0) = y$



$$M_y(x) := \frac{y - P_y x - sQ_y x}{1 - \langle x, y \rangle}, \quad x \in \mathbf{B},$$

$$P_0(x) \equiv 0 \qquad P_y(x) := \frac{\langle x, y \rangle}{\|y\|^2} y, \quad y \neq 0$$

$$Q_y = I - P_y \qquad s = \sqrt{1 - \|y\|^2} \qquad M_y^{-1} = M_y$$

$$U_y \tau = M_y \tau \qquad m := M_y \circ U_y$$

M.Elin, S.Reich and D.Shoikhet, Dissertationes Math., 2004

$f_m(w) := m'(w)^{-1} f(m(w)), \quad w \in \mathbf{B}$ is a generator on $\mathbf{B}$

$g(z) := \langle f_m(z\tau), \tau \rangle, \quad z \in \Delta$ is a generator on Δ

$g(r) := \left\langle [m'(r\tau)]^{-1} f(m(r\tau)), \tau \right\rangle, \quad r \in [0,1)$

$\Gamma : [0,1) \mapsto \mathbf{B}, \quad \Gamma(r) := m(r\tau), \quad 0 \le r < 1$ is a restricted curve:

1. $\lim\limits_{t \to 1^-} \dfrac{\|\Gamma(t) - \gamma(t)\|^2}{1 - \|\gamma(t)\|^2} = 0$

$$\frac{\|m(r\tau) - \langle m(r\tau), \tau \rangle \tau\|^2}{1 - |\langle m(r\tau), \tau \rangle|^2} = \frac{(1-r)\left(\|y\|^2 - |\langle y, \tau \rangle|^2\right)}{(1-r)\left(2\|y\|^2 - 1 - |\langle y, \tau \rangle|^2\right) + 2\left(1 - \|y\|^2\right)} \underset{r \to 1^-}{\longrightarrow} 0$$

2. $$\frac{\|\gamma(t)-\tau\|}{1-\|\gamma(t)\|} \le A < \infty \quad (0 \le t < 1)$$

$$\frac{\|\tau-\langle m(r\tau),\tau\rangle\tau\|}{1-\|\langle m(r\tau),\tau\rangle\tau\|} \le \frac{|1-\langle m(r\tau),\tau\rangle|}{1-\|m(r\tau)\|} = \frac{|1-\langle m(r\tau),\tau\rangle|^2}{1-\|m(r\tau)\|^2}\cdot\frac{1+\|m(r\tau)\|}{|1-\langle m(r\tau),\tau\rangle|}$$

$$\frac{|1-\langle m(r\tau),\tau\rangle|^2}{1-\|m(r\tau)\|^2} = L\frac{1-r}{1+r},$$

where $$L := \frac{d}{dz}\langle m(z\tau),\tau\rangle\Big|_{z=1} = \frac{|1-\langle y,\tau\rangle|^2}{1-\|y\|^2} > 0.$$

$$\frac{\|\tau-\langle m(r\tau),\tau\rangle\tau\|}{1-\|\langle m(r\tau),\tau\rangle\tau\|} \le L\frac{1-r}{1+r}\cdot\frac{1+\|m(r\tau)\|}{|1-\langle m(r\tau),\tau\rangle|} < 2L\frac{1}{\left|\frac{1-\langle m(r\tau),\tau\rangle}{1-r}\right|} \underset{r\to 1^-}{\longrightarrow} 2$$

$$g(r) := \left\langle [m'(r\tau)]^{-1} f(m(r\tau)), \tau \right\rangle, \quad r \in [0,1)$$

$$\frac{g(r)}{(1-r)^3} = \frac{\|m(r\tau)-\tau\|^3}{(1-r)^3} \left\langle \frac{f(m(r\tau))}{\|m(r\tau)-\tau\|^3}, \left[M'_y(m(r\tau))\right]^* U_y \tau \right\rangle$$

The **weak restricted K-limit** of $\dfrac{f(x)}{\|x-\tau\|^3}$ at $\tau \in \partial\mathbf{B}$

equals 0 $\quad \Rightarrow \quad$ for each $u \in H$,

$$\left\langle \frac{f(x)}{\|x-\tau\|}, u \right\rangle \to 0$$

as x tends to τ along every restricted curve ending at τ .

$$\Rightarrow \quad \lim_{r \to 1^-} \frac{g(r)}{(1-r)^3} = 0 \quad \underset{Theorem\ 1}{\Rightarrow} \quad g(z) \equiv 0 \text{ on } \Delta$$

$$\left\langle [m'(z\tau)]^{-1} f(m(z\tau)), \tau \right\rangle = 0, \quad \forall z \in \Delta$$

$z = 0:$

$$\left\langle [m'(0)]^{-1} f(y), \tau \right\rangle = 0, \quad \forall y \in \mathbf{B}$$

$$\left\langle f(y), y - \tau + \|y\|^2 \tau - \langle \tau, y \rangle y \right\rangle = 0, \quad \forall y \in \mathbf{B}$$

Show: $\left\langle f(y), x \right\rangle = 0, \quad \forall x \in H$

$$\begin{cases} y = y_1 \tau + \tilde{y} \\ f(y) = f_1(y)\tau + \tilde{f}(y) \end{cases}$$

$$\left(1 - \overline{y}_1 - \|\tilde{y}\|^2\right) f_1(y) = (1 - y_1)\left\langle \tilde{f}(y), \tilde{y} \right\rangle \quad \forall y \in \mathbf{B}$$

$\dfrac{d}{d\overline{y}_1}:$ 1). $f_1(y) = 0 \quad \forall y \in \mathbf{B}$

2). $\left\langle \tilde{f}(y), \tilde{y} \right\rangle = 0 \quad \forall y \in \mathbf{B}$

$$\left\langle \tilde{f}(y), \tilde{y} \right\rangle = 0 \quad \forall y \in \mathbf{B}$$

$$\left\langle f(y), \tilde{y} \right\rangle = 0 \quad \forall y \in \mathbf{B}$$

Let σ be a unit vector orthogonal to $\tau : \langle \sigma, \tau \rangle = 0, \ \|\sigma\| = 1$

$$\begin{cases} \tilde{y} = y_2 \sigma + u \\ f(y) = f_2(y)\sigma + v(y) \end{cases}$$

$$f_2(y)\overline{y}_2 = -\langle v(y), u \rangle \quad \forall y \in \mathbf{B}$$

$$\frac{d}{d\overline{y}_2}: \qquad f_2(y) = 0 \quad \forall y \in \mathbf{B}$$

$$f(y) \equiv 0 \quad \forall y \in \mathbf{B}$$