

A gentle introduction to Schwinger's formulation of quantum mechanics: The groupoid picture

F. M. Ciaglia

*Dipartimento di Fisica "E. Pancini", Università di Napoli Federico II,
Complesso Universitario di Monte S. Angelo Edificio 6, Napoli 80126, Italy
INFN-Sezione di Napoli, Complesso Universitario di Monte S. Angelo Edificio 6,
Napoli 80126, Italy
florio.m.ciaglia@gmail.com; ciaglia@na.infn.it*

A. Ibert

*ICMAT, Instituto de Ciencias Matemáticas (CSIC-UAM-UC3M-UCM)
Depto. de Matemáticas, Univ. Carlos III de Madrid, Leganés, Madrid 28911, Spain
albertoi@math.uc3m.es*

G. Marmo

*Dipartimento di Fisica "E. Pancini", Università di Napoli Federico II,
Complesso Universitario di Monte S. Angelo Edificio 6, Napoli 80126, Italy
INFN-Sezione di Napoli, Complesso Universitario di Monte S. Angelo Edificio 6,
Napoli 80126, Italy
marmo@na.infn.it*

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In this paper, we review Schwinger's formulation of Quantum Mechanics and argue that the mathematical structure behind Schwinger's "Symbolism of Atomic Measurements" is that of a groupoid. In this framework, both the Hilbert space (Schrödinger picture) and the C^* -algebra (Heisenberg picture) of the system turn out to be derived concepts, that is, they arise from the underlying groupoid structure.

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1. Introduction

At the dawn of quantum mechanics, two pictures for the description of quantum systems emerged. One, proposed by Heisenberg, was called matrix mechanics and

Corresponding author

kept the form of classical Hamiltonian equations for noncommutative quantities, the other, proposed by Schrödinger, was named wave mechanics and was formulated in analogy with wave equations. In both of them was clearly present what Dirac called the “principle of analogy”. It may be worth mentioning that Dirac in his lectures on Quantum Field Theory,⁴ talking about Basic Quantum Concepts, clearly says: “what is the precise mathematical nature of the dynamical variables? It is better, for the present, to keep an open mind about these dynamical variables and just call them q -numbers”. He then goes on by imposing algebraic axioms that look reasonable, postponing the problem of “identification as physical entities”.

Few years later, searching for a relativistically covariant description of quantum systems, Dirac proposed a Lagrangian approach to the description of quantum systems.² This few-pages paper inspired, separately, both Feynman and Schwinger to provide an alternative formulation, or “picture”, of quantum mechanics by means of a Lagrangian or an “action principle”, respectively. It is quite interesting to note that when addressing the problem of introducing quantum mechanics at an elementary level (text-book level), both Feynman and Schwinger made recourse to the Stern–Gerlach experiment. Feynman used it in his presentation of quantum mechanics in the third volume of his famous lectures, and Schwinger used it in his formulation of quantum mechanics in terms of “Symbolism of Atomic Measurements”.⁹

Now, the primary mathematical object for the “Schrödinger picture” was identified by Dirac in the Hilbert space of the system,³ and states, observables and evolution, along with a probability function, were identified as objects built out of the Hilbert space of the system. While, for the Heisenberg picture, the primary mathematical structure was identified to be a C^* -algebra assigned to each quantum system, and observables, states, evolution and probability function were clearly identified by von Neumann.¹⁰ On the other hand, it seems to us that the mathematical structure underlying Schwinger’s picture of quantum mechanics has not yet been identified in its bare essentials. A thorough extensive presentation will be published elsewhere.¹ In this short note, we would like to argue that the proper mathematical structure emerging from the “Symbolism of Atomic Measurements” is that of a groupoid and its higher algebraic generalizations. In some sense, we could argue that in this picture, the probabilistic point of view plays a much more prominent role than in the other two pictures attributed, respectively, to Schrödinger and Heisenberg. In this aspect, Schwinger’s proposal is closer to the positivistic thinking of Heisenberg.

2. Symbolism of Atomic Measurements and Groupoids

In the incipit of his first note,⁸ Schwinger writes: “The classical theory of measurement is implicitly based upon the concept of an interaction between the system of interest and the measuring apparatus that can be made arbitrarily small, or at least precisely compensated, so that one can speak meaningfully of an idealized

experiment that disturbs no property of the system. The classical representation of physical quantities by numbers is the identification of all properties with the results of such non-disturbing measurements. It is characteristic of atomic phenomena, however, that the interaction between system and instrument cannot be indefinitely weakened. Nor can the disturbance produced by the interaction be compensated precisely since it is only statistically predictable. Accordingly, a measurement on one property can produce unavoidable changes in the value previously assigned to another property, and it is without meaning to ascribe numerical values to all the attributes of a microscopic system. The mathematical language that is appropriate to the atomic domain is found in the *symbolic transcription of the laws of microscopic measurement.*^a

Moved by this operational attitude, Schwinger considers idealized physical systems described by means of a maximal set^b \mathbf{A} of compatible observables presenting only a finite number of possible outcomes of measurements. These possible outcomes are denoted as a, a', \dots, a^n . At this point, the so-called measurement symbols $M(a, a')$ are introduced without any reference to the Hilbert space and without making recourse to any analogy with classical physics. In Schwinger's own words,⁸ the measurement symbol " $M(a', a)$ indicates a selective measurement in which systems are accepted only in the state a and emerge in the state a' ". In particular, the case in which no change occurs is denoted with $M(a) \equiv M(a, a)$. The next step consists in giving a composition law for measurement symbols:

$$M(a''', a'')M(a', a) = \delta(a', a'')M(a''', a), \quad (1)$$

where δ is the Kronecker delta. This composition of measurement symbols is postulated and it reflects the fact that when successive measurements are considered, the second stage $M(a''', a'')$ of the full measurement scheme accepts the output of the first stage $M(a', a)$ if and only if $a' = a''$. Quite interestingly, reversing the order in the compound measurement scheme, we obtain $M(a', a)M(a''', a'') = \delta(a, a''')M(a', a'')$ which means that the composition of measurement symbols is noncommutative.

To make contact with the well-known Dirac–Schrödinger formalism, let us consider the association of a Hilbert space with every quantum system, and use the bra-ket formalism.^c In this setting, Schwinger's maximal set of compatible observables translates into the notion of maximal set of mutually commuting self-adjoint operators on the Hilbert space of the system. In the finite-dimensional case, we may consider a single observable (self-adjoint operator) \mathbf{A} whose spectrum is necessarily discrete, say $\{a_1, \dots, a_n\}$, and we obtain a resolution of the identity in terms of an orthonormal basis of eigenvectors, that is, $1 = \sum_j |a_j\rangle\langle a_j|$ with $\langle a_j|a_k\rangle = \delta_{jk}$. It is

^aThe emphasizing is due to the authors.

^bIn the general picture, discussed in Ref. 1, we do not require maximality. Such an approach is much more flexible and allows notions like Sorkin's "events"⁶ to be considered in the same ground.

^cIn the abstract groupoid formalism of Ref. 1, the Hilbert space is provided gratis as it comes from the fundamental representation of the groupoid.

then clear that the elements $M(a_j, a_k) = |a_j\rangle\langle a_k|$ represent Schwinger’s measurement symbols.

When endowed with the associative product inherited from linear operators, the set Γ_A of such symbols forms a groupoid, more precisely, a pair-groupoid.⁷ Indeed, for every $M(a_j, a_k) \in \Gamma_A$ we can define the source and target maps $s, t: \Gamma_A \rightarrow \Gamma_A^0$, where $\Gamma_A^0: \{M(a_j, a_j) \in \Gamma_A\}$, given by $s(M(a_j, a_k)) = M(a_k, a_k)$ and $t(M(a_j, a_k)) = M(a_j, a_j)$, and it is important to immediately check that the (associative) composition law $M(a_j, a_k) \circ M(a_l, a_m)$ coming from the operator product between $M(a_j, a_k) = |a_j\rangle\langle a_k|$ and $M(a_l, a_m) = |a_l\rangle\langle a_m|$ makes sense whenever $s(M(a_j, a_k)) = t(M(a_l, a_m))$, in which case it is $M(a_j, a_k) \circ M(a_l, a_m) = M(a_j, a_m)$. Furthermore, every $M(a_j, a_k)$ has a (right and left) inverse $M^{-1}(a_j, a_k) = M(a_k, a_j)$.

Note that, the abstract composition law for measurement symbols is postulated by Schwinger, while here it appears arising from operators simply because of our choice of representation. In a forthcoming work,¹ we will take up the abstract point of view in order to reformulate Schwinger’s ideas in terms of a purely abstract groupoidal picture of quantum mechanics in which the Hilbert space and the C^* -algebra of (bounded) observables will be “derived concepts”.

The full algebra of quantum measurements introduced by Schwinger is nothing but the “groupoid-algebra” associated with Γ_A , which is the analogue of the well-known “group-algebra” in group theory with the corresponding “convolution product”. In the axiomatics presented by Schwinger, the “groupoid” and its “groupoid algebra”¹¹ are not distinguished, they are introduced at the same time. The “objects” are identified with the unities and the Hilbert space arises from the fundamental representation. In the bra-ket notation, the groupoid-algebra is given by complex linear combinations of $|a_j\rangle\langle a_k|$. An involution \dagger on the groupoid-algebra is immediately defined starting from $M^\dagger(a_j, a_k) = (|a_j\rangle\langle a_k|)^\dagger = |a_k\rangle\langle a_j| = M(a_k, a_j)$ and extending it by linearity. Real elements will be generated by $(|a_j\rangle\langle a_k| + |a_k\rangle\langle a_j|)$ and $i(|a_j\rangle\langle a_k| - |a_k\rangle\langle a_j|)$ with real coefficients, and we already know that there is a unit element $\sum_j |a_j\rangle\langle a_j| = \mathbb{1}$. When \mathcal{H} is finite-dimensional, it is clear that we recover the full C^* -algebra of bounded linear operators on \mathcal{H} . In the infinite-dimensional case, some care must be taken in order to overcome convergence issues of infinite sums.

Schwinger’s immediately recognized that “the physical quantities contained in one complete set **A** do not comprise the totality of physical attributes of the system. One can form other complete sets, **B**, **C** and so on, which are mutually incompatible, and for each choice of noninterfering physical characteristics, there is a set of selective measurements referring to systems in the appropriate states”.⁸ From the probabilistic point of view, this means that the groupoid Γ_A is not enough to encode all the information on the quantum state of the system. Indeed, in the Hilbert space formulation of quantum mechanics, a (pure) state is represented by a “ray”, that is, a “one-dimensional subspace” of \mathcal{H} , associated with non-null vector

$|\psi\rangle \in \mathcal{H}$, and when we select a maximal set of compatible observables (resolution of the identity), we are able to build a probability vector out of $|\psi\rangle$ by setting

$$p_j(\psi) := \frac{\langle a_j|\psi\rangle\langle\psi|a_j\rangle}{\langle\psi|\psi\rangle}, \quad (2)$$

so that $p_j(\psi) \geq 0$ and $\sum_j p_j(\psi) = 1$. Clearly, this single probability distribution is not enough to reconstruct the quantum state from which we started. This is clearly illustrated in the so-called tomographic picture of quantum mechanics,⁷ where fair probability distributions, tomograms, are associated with every quantum state, and it is proved that every quantum state may be “reconstructed” when a sufficient set, a “quorum”, of tomograms has been provided.⁵ In fact, considering another resolution of the identity $\mathbb{1} = \sum_k |b_k\rangle\langle b_k|$, a new probability vector is associated with $|\psi\rangle$

$$p'_k(\psi) := \frac{\langle b_k|\psi\rangle\langle\psi|b_k\rangle}{\langle\psi|\psi\rangle} \quad (3)$$

and it is then clear that all the information regarding the possible probability distributions is encoded in the abstract object $\rho = \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle}$ which is independent of the specific resolution of the identity used to define a probability distribution. We may call this object the “abstract probability amplitude”. The replacement of classical fair probability distributions by quantum probability amplitudes is a crucial ingredient to be able to describe such quantum phenomena as interference and entanglement.

We may say that the existence of maximal sets of compatible observables that are mutually incompatible leads to the transition from “probability distributions” to “probability amplitudes”.

Referring again to the bra-ket formalism, by choosing a different observable, say $\mathbf{B} = \sum_k b_k |b_k\rangle\langle b_k|$, we obtain another resolution of the identity $\mathbb{1} = \sum_k |b_k\rangle\langle b_k|$, and we can build another pair-groupoid Γ_B whose elements are $M(b_k, b_l) = |b_k\rangle\langle b_l|$, and thus another groupoid-algebra associated with Γ_B . By introducing $U_{AB} := \sum_k |a_k\rangle\langle b_k|$ and noting that $U_{AB}^\dagger = U_{BA} = \sum_k |b_k\rangle\langle a_k|$, we may build a unitary map that “transforms” the groupoid Γ_B into the groupoid Γ_A acting on every $M(b_j, b_k) \in \Gamma_B$ by conjugation with U_{AB} , that is, $M(b_j, b_k) \mapsto U_{AB}M(b_j, b_k)U_{AB}^\dagger = M(a_j, a_k)$. From this, it easily follows that, in the finite-dimensional case, Γ_A and Γ_B are isomorphic as groupoids, and thus, their groupoid-algebras are isomorphic as C^* -algebras.

In Schwinger’s work, measurement symbols pertaining to different (incompatible) sets of compatible observables are considered and denoted with the formal symbols $M(a, b)$. These symbols have to encode the fact that “measurements of properties \mathbf{B} , performed on a system in a state c that refers to properties incompatible with \mathbf{B} , will yield a statistical distribution of the possible values. Hence, only a determinate fraction of the systems emerging from the first stage will be accepted by the second stage”, and thus the following composition law is introduced:

$$M(a, b)M(c, d) = \langle b, c\rangle M(a, d), \quad (4)$$

where $\langle b, c \rangle$ is a number characterizing the above-mentioned “statistical distribution”. For instance, when the measurement symbols considered pertain to the same maximal set of compatible observables, that is, $b = a$ and $c = a'$, we have $\langle a, a' \rangle = \delta(a, a')$. It is important to introduce the formal symbols $M(a, b)$ because, roughly speaking, they can be put into correspondence with composition of Stern–Gerlach devices and they uncover a more profound mathematical structure underlying Schwinger’s formalism of atomic measurements which is that of a 2-groupoid. We will comment thoroughly on this subject in a forthcoming work.¹

In the bra-ket formalism, we may consider the symbols $M(a_j, a_k) = |a_j\rangle\langle a_k|$ with $M(b_l, b_m) = |b_l\rangle\langle b_m|$ associated with different resolution of the identity, and we may compose them by means of the operator product between $|a_j\rangle\langle a_k|$ and $|b_l\rangle\langle b_m|$ obtaining

$$M(a_j, a_k)M(b_l, b_m) = |a_j\rangle\langle a_k| |b_l\rangle\langle b_m| = \langle a_k|b_l\rangle M(a_j, b_m). \quad (5)$$

It is clear that the quantum mechanical transition probability amplitude $\langle a_k|b_l\rangle$ plays the role of the statistical coefficient $\langle a, b \rangle$ introduced by Schwinger. Indeed, remaining in the probabilistic setting, we may review what Schwinger did in terms of abstract “transition probability amplitudes”. Indeed, introducing the resolution of the identity $1 = \sum_j |a_j\rangle\langle a_j|$, we may write the transition probability

$\left| \frac{\langle \psi|\varphi \rangle}{\sqrt{\langle \psi|\psi \rangle}\sqrt{\langle \varphi|\varphi \rangle}} \right|^2$ between pure quantum states as

$$p(\psi, \varphi) = \left| \frac{\langle \psi|\varphi \rangle}{\sqrt{\langle \psi|\psi \rangle}\sqrt{\langle \varphi|\varphi \rangle}} \right|^2 = \sum_{j,k} \frac{\langle a_j|\psi \rangle \langle \varphi|a_j \rangle \langle a_k|\varphi \rangle \langle \psi|a_k \rangle}{\langle \psi|\psi \rangle \langle \varphi|\varphi \rangle} \quad (6)$$

and similarly we would obtain $p'(\psi, \varphi)$ if we change basis. Again, the abstract mathematical object we would obtain if we get rid of the particular basis would be either

$$T(\psi, \varphi) := \frac{|\psi\rangle\langle \varphi|}{\sqrt{\langle \psi|\psi \rangle}\sqrt{\langle \varphi|\varphi \rangle}} \quad \text{or} \quad T(\varphi, \psi) := \frac{|\varphi\rangle\langle \psi|}{\sqrt{\langle \psi|\psi \rangle}\sqrt{\langle \varphi|\varphi \rangle}}. \quad (7)$$

These two different objects will give rise to the same “transition probability” when a basis for the Hilbert space is considered. We may call these objects “abstract transition probability amplitudes”.

When we consider two (incompatible) maximal sets of compatible observables \mathbf{A} and \mathbf{B} , we may form the matrix of transition probabilities between the two resolutions of the identity as the matrix whose entries are given by

$$p(a_j, b_k) = \langle a_j|b_k\rangle\langle b_k|a_j\rangle. \quad (8)$$

In Schwinger’s terminology,⁸ this matrix is the “transformation function” relating the \mathbf{A} -description with the \mathbf{B} -description. It is a stochastic linear map because $p(a_j, b_k) \geq 0$ and $\sum_j p(a_j, b_k) = 1$. Furthermore, as $p(a_j, b_k) = \langle a_j|b_k\rangle\langle b_k|a_j\rangle = \langle b_k|a_j\rangle\langle a_j|b_k\rangle = p(b_k, a_j)$, the matrix is symmetric and consequently the associated map is doubly stochastic.

It should be remarked that it is possible to consider pairs of *complementary observables* such that $p(a_j, b_k)$ is independent of the specific a_j, b_k and would be equal to $\frac{1}{n}$, if n is the number of different outcomes. The interpretation is that if the system is prepared in a state in which the value of **A** can be predicted with certainty, then all measurement results are equally probable in a measurement of **B**. This feature of quantum mechanics has no true analog in classical physics.

3. Conclusions

In conclusion, we have seen how the notion of groupoid naturally emerges when one considers Schwinger's Symbolism of Atomic Measurements underlying an algebra that turns out to be a specific representation of the corresponding groupoid-algebra. This groupoid structure is seen to emerge when we replace transition probabilities with transition amplitudes. Roughly speaking, this corresponds to passing from $\langle\psi|\varphi\rangle$ to $|\psi\rangle\langle\varphi|$ or $|\varphi\rangle\langle\psi|$. We have seen how an abstract groupoid structure may be identified in the Hilbert space formalism of quantum mechanics by using an orthonormal discrete resolution of the identity $1 = \sum_j |a_j\rangle\langle a_j|$ and defining the elements of the groupoid as $M(a_j, a_k) := |a_j\rangle\langle a_k|$. By using different resolutions of the identity, we may build different groupoids that, in the finite-dimensional case, turn out to be isomorphic. Once we have the groupoid, we are able to build its groupoid-algebra which is naturally an involution algebra that, in the finite-dimensional case, coincides with the full C^* -algebra of the bounded linear operators on the Hilbert space of the system, thus giving a first insight regarding the connection between the groupoid picture of quantum mechanics presented here and the C^* -algebraic picture of quantum mechanics generalizing Heisenberg's picture. All these considerations will be thoroughly examined in a forthcoming paper¹ where the groupoid picture will be taken as the starting point for a formulation of a quantum theory, and the Hilbert space and C^* -algebra formalisms will be recovered from the groupoid structure alone.

As Schwinger says:⁸ "The entire discussion remains restricted to the realm of quantum statics which, in its lack of explicit reference to time, is concerned either with idealized systems such that all properties are unchanged in time or with measurements performed at a common time". Evolution in time as well as the statistical interpretation of the theory, will be taken up elsewhere and its formalization constitutes the essence of "Schwinger variational principle".

In conclusion, while in the Schrödinger picture we associate a Hilbert space with every quantum system, a C^* -algebra in the Heisenberg picture, we could say that we associate a groupoid with every quantum system in the Schwinger picture.

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