SCHWINGER'S PICTURE OF QUANTUM MECHANICS: ALGEBRAS AND OBSERVABLES[1]

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Abstract. The kinematical foundations of a new picture of Quantum Mechanics based on the theory of groupoids was presented in [1]. This groupoids based picture provides the mathematical background for Schwinger's algebra of selective measurements and his quantum variational principle. Category theory, in particular the notion of 2-groupoids as well as their representations, is used in the description of the new picture.

In this paper the dynamical aspects of the theory are analysed as well as its statistical interpretation. For that, the algebra generated by the observables as well as the notion of states are analysed and the structure of transition functions, that play an instrumental role in Schwinger's picture, are elucidated. A Hamiltonian picture of dynamical evolution emerges naturally and the formalism offers a simple way to discuss the quantum-to-classical transition. Some basic examples are examined and the relation with the standard Dirac-Schrödinger and Born-Jordan-Heisenberg pictures are discussed.

Contents

[1]THIS IS A DRAFT. NOT THE FINAL VERSION YET.

1. Introduction: Groupoids and quantum systems

In the previous work [1], following the insight provided by Schwinger's picture of Quantum Mechanics [2, 3], it was argued that the basic mathematical structure to describe a physical system is a 2-groupoid.

Schwinger's algebra of measurements, his foundational approach to describe quantum systems and quantized fields, is based on the notion of selective and compound measurements [3]. Based on that, Schwinger developed a theory of transitions functions that, together with a dynamical principle, set the basis to his solution of the quantum description of electrodynamics (see the series of celebrated papers [4]).

After a careful analysis of Schwinger's algebra of measurements it was argued in [1] that the abstract description of quantum mechanical systems should be formulated in terms of a family of primary notions: 'events', corresponding to elementary selective measurements; 'transitions', that in Schwinger's simplified presentation were called generalised selective measurements, and 'transformations', that were used to compare descriptions corresponding to different incompatible experimental setups.

The structural properties of such notions were discussed at length and it was shown that they have the mathematical structure known as a 2-groupoid. In fact, events and transitions provide a natural abstract setting for Schwinger's notion of physical selective measurements and form an ordinary groupoid. The theory of transformations fits naturally in this setting and determines a 2-groupoid structure on top of Schwinger's groupoid, the groupoid defined by the transitions of the system and its corresponding objects, the events.

The description of the mathematical structure behind Schwinger's algebra of measurements provided in [1] was essentially kinematical and no attention was paid to the dynamical aspects of the theory. Thus, it can be considered as a background structure for any quantum mechanical system. Only the broad aspects of the theory, like the general form of events (but not their quantitative characteristics), the relations among them, with its categorical trait, and the inner symmetries in the form of transformations, were accounted for at this stage. It was shown that the fundamental representation of Schwinger's groupoid algebra allows to relate the groupoid picture to Dirac's picture of Quantum Mechanics by associating a Hilbert space to it, again reinforcing this kinematical interpretation as no dynamics in the form of a Hamiltonian operator is specified[1]. Thus, an analysis of the fundamental dynamical aspects of the theory, starting with the notion of observable and states, should complement the work in [1]. This will be main objective of the present paper.

Here we would like to discuss in detail the role of dynamical variables, that is, observables, and the dynamical evolutions in the groupoid setting. Observables will be defined in terms of the basic notion of amplitudes. An 'amplitude' would be defined as the assignment of a complex numerical value to any physically allowed transition of the system. Thus, amplitudes are just complex valued functions on Schwinger's groupoid and they carry a C^*-algebra structure. The physical observables are then the real elements in this C^*-algebra.

A complete description of the system will be provided by a groupoid such that the real elements in its algebra of amplitudes are actually the totality of observables of the theory. In such case, the states of theory are the states of the C^*-algebra of amplitudes, and their relation with vectors in the fundamental representation of the groupoid will be discussed by means of the GNS construction. The the standard probabilistic interpretation of the theory can be established by means of the module square of amplitudes of the operators representing the observables.

The many different, but equivalent, descriptions of the same physical system provided by (mutually incompatible) different complete families of experimental setups allow to introduce a large class of generalised transitions, called in this paper Stern-Gerlach transitions, which provide the mathematical background for Schwinger's theory of transitions functions and open the path towards the formulation of a genuine dynamical principle for quantum systems. Some basic properties of transition functions and their dynamical properties will be analysed, however, the discussion of Schwinger's dynamical principle and its subsequent applications will be discussed elsewhere [5].

Before starting the actual presentation of the ideas sketched before, it is worth to devote a few lines to place the aim and scope of the present project among the many existing approaches regarding the foundations of Quantum Mechanics that could be related to it.

Apart from the standard well-known pictures of Quantum Mechanics already discussed in [1], many other settings have been proposed, some of them motivated by the problem of achieving a quantum theoretical description of Gravity. Without pretending to be exhaustive, not even covering all relevant contributions on the subject, we would like to mention here R. Penrose's spin-networks [6], [7], von Weizsacker *urs* [8], [9], the theory of *causalnets* developed from R. Sorkin's insight [10, 11], C. Isham's categorical foundation of gravity [12], the noncommutative geometry approach to the description of space-time inspired on A. Connes conception of geometry [13], [14], [15], etc. All of them share a notion of "discretness"

[1]Note that all infinite-dimensional separable Hilbert spaces are isometrically isomorphic, thus, they do not provide a distinction between quantum systems.

and "non-commutativity" in the description of fundamental physical theories in Dirac's spirit [16, 17]. Even if we will not offer here a proper analysis of the relation of the present discussion with any of them, we may state that the groupoid description distilled from Schwinger's ideas is related to all of them as it describes physical systems without recurring to any notion of space-time; moreover, this description incorporates in a natural way a statistical interpretation and may naturally account for non-commutativity. However, we must stress here that we do not pretend to use it as an alternative foundation for a 'quantum' theory of gravity.

The paper will be organised as follows. We will start by succinctly reviewing the basic notions and notations used in our previous work and, afterwards, we will discuss the properties and structure of the algebra of observables of the theory. The notion of a complete description of a physical system will be introduced and the C^*-structure of the algebra of observables will be discussed. The notion of states and the construction of the corresponding vector descriptions in terms of the fundamental representation of the groupoid algebra will be presented by using the GNS construction. It will be shown that Schwinger's transition functions are naturally described in this setting and a discussion of the properties of transition functions will be offered. Finally, the construction of the dynamical evolution of closed systems will be analysed proving that a Hamiltonian observable must be the infinitesimal generator of it. Then, we will end the paper by applying all the previous ideas to discuss a few simple systems: the qubit and the harmonic oscillator. These examples, even if elementary, illustrate the powerful analytical insight offered by the groupoid approach.

As it was commented before, the discussion of Schwinger's dynamical principle as well as a detailed description of the probabilistic interpretation of the theory in terms of Sorkin's quantum measures [11], as well as the application to other physical systems of interest, will be left for subsequent works.

2. Groupoids, algebras and other basic notions

Even if groupoids can be described in a very abstract setting using category theory, in this paper we will only use set-theoretical concepts and notations to work with them. Thus, a groupoid $\mathbf{G}$ will be a set whose elements α will be called transitions. There are two maps $s, t \colon \mathbf{G} \to \Omega$, called source and target respectively, from the groupoid $\mathbf{G}$ into a set Ω whose elements will be called events, and, if $s(\alpha) = a$ and $t(\alpha) = a'$, we will often use the diagrammatic representation $\alpha \colon a \to a'$ for the transition α. Notice that the previous notation does not imply that α is a map from a set a into another set a', even if sometimes we will use the notation $\alpha(a)$ to denote $a' = t(\alpha)$. We will also say that the transitions α relates the event a to the event a'.

Denoting by $\mathbf{G}(a, a')$ the set of transitions relating the event a with the event a', there is a composition law $\circ \colon \mathbf{G}(a', a'') \times \mathbf{G}(a, a') \to \mathbf{G}(a, a'')$ such that if $\alpha \colon a \to a'$

and $\beta\colon a' \to a''$, then $\beta \circ \alpha\colon a \to a''$[2]. It is postulated that the composition law $\circ$ is associative whenever the composition of three transitions makes sense, that is: $\gamma \circ (\beta \circ \alpha) = (\gamma \circ \beta) \circ \alpha$, whenever $\alpha\colon a \to a'$, $\beta\colon a' \to a''$ and $\gamma\colon a'' \to a'''$. For any event $a \in \Omega$ there is a transition denoted by 1_a satisfying the properties $\alpha \circ 1_a = \alpha$, $1_{a'} \circ \alpha = \alpha$ for any $\alpha\colon a \to a'$. Notice that the assignment $a \mapsto 1_a$ defines a natural inclusion $i\colon \Omega \to \mathbf{G}$ of the space of events in the groupoid $\mathbf{G}$. Finally it will be assumed that any transition $\alpha\colon a \to a'$ has an inverse, that is there exists $\alpha^{-1}\colon a' \to a$ such that $\alpha \circ \alpha^{-1} = 1_{a'}$, and $\alpha^{-1} \circ \alpha = 1_a$.

Given an event $a \in \Omega$, we will denote by $\mathbf{G}_+(a)$ the set of transitions starting at a, that is, $\mathbf{G}_+(a) = \{\alpha\colon a \to a'\} = s^{-1}(a)$. In the same way we define $\mathbf{G}_-(a)$ as the set of transitions ending at a, that is, $\mathbf{G}_-(a) = \{\alpha\colon a' \to a\} = t^{-1}(a)$. The intersection of $\mathbf{G}_+(a)$ and $\mathbf{G}_-(a)$ is the set of transitions starting and ending at a and is called the isotropy group G_a at a: $G_a = \mathbf{G}_+(a) \cap \mathbf{G}_-(a)$. Notice that we may write

$$\mathbf{G} \circ 1_a = \mathbf{G}_+(a)\,, \qquad 1_a \circ \mathbf{G} = \mathbf{G}_-(a)\,, \tag{1}$$

in the sense that composing with the unit 1_a on the right selects the transitions starting at a. Indeed, a transition α which is the result of composing some other transition with 1_a must have its source at a. In fact, it is easy to check that $\mathbf{G} \circ \alpha = \mathbf{G}_+(s(\alpha))$ and $\alpha \circ \mathbf{G} = \mathbf{G}_-(t(\alpha))$.

Given an event a, the orbit $\mathcal{O}_a$ of a is the subset of all events related to a, that is, $a' \in \mathcal{O}_a$ if there exists $\alpha\colon a \to a'$. Clearly the isotropy group G_a acts on the right on the space of transitions leaving from a, that is, there is a natural map $\mu_a\colon \mathbf{G}_+(a) \times G_a \to \mathbf{G}_+(a)$, given by $\mu_a(\alpha, \gamma_a) = \alpha \circ \gamma_a$ (notice that the transition $\gamma_a\colon a \to a$ doesn't change the source of $\alpha\colon a \to a'$). Then it is easy to check that there is a natural bijection between the space of orbits of G_a in $\mathbf{G}_+(a)$ and the elements in the orbit $\mathcal{O}_a$, given by $\alpha \circ G_a \mapsto \alpha(a) = a'$. Then we may write:

$$\mathbf{G}_+(a)/G_a \cong \mathcal{O}_a\,.$$

It is obvious that there is also a natural left action of G_a into $\mathbf{G}_-(a)$ and that $G_a \backslash \mathbf{G}_-(a) \cong \mathcal{O}_a$ too. The subset $\mathbf{G}_+(a)$ is left-invariant under the natural action of the groupoid $\mathbf{G}$ on it, that is $\mathbf{G} \circ \mathbf{G}_+(a) = \mathbf{G}_+(a)$. In the same way $\mathbf{G}_-(a)$ is right invariant under the action of $\mathbf{G}$. Notice that $\mathbf{G} \circ \mathbf{G}_-(a) = \mathbf{G}(a) = \mathbf{G}_+(a) \circ \mathbf{G}$, in fact, because of (1), we have:

$$\mathbf{G} \circ 1_a \circ \mathbf{G} = \mathbf{G}(a)\,. \tag{2}$$

The groupoid algebra $\mathbb{C}[\mathbf{G}]$ of the groupoid $\mathbf{G}$ is defined in the standard way as the associative algebra generated by the elements of $\mathbf{G}$ with the relations provided by the composition law of the groupoid, that is, elements $\boldsymbol{\alpha}$ in $\mathbb{C}[\mathbf{G}]$ are finite formal linear combinations $\boldsymbol{\alpha} = \sum_{\alpha \in \mathbf{G}} c_\alpha\, \alpha$, with c_α complex numbers. The groupoid

[2] The 'backwards' notation for the composition law has been chosen so that the various representations and compositions used along the paper look more natural, it is also in agreement with the standard notation for the composition of functions.

algebra elements $\boldsymbol{\alpha}$ can be though as mixed transitions for the system. Once we introduce the C^*-algebra of amplitudes in the groupoid picture, the convex combinations of the unit transitions 1_a with $a \in \Omega$ may be thought of as the normal states of the algebra of amplitudes. The associative composition law on $\mathbb{C}[\mathbf{G}]$ is defined as:

$$\boldsymbol{\alpha} \cdot \boldsymbol{\alpha}' = \sum_{\alpha,\alpha' \in \mathbf{G}} c_\alpha c_{\alpha'} \, \delta_{\alpha,\alpha'} \; \alpha \circ \alpha' \, ,$$

where the indicator function $\delta_{\alpha,\alpha'}$ takes the value 1 if α and α' are composable, and zero otherwise. The groupoid algebra has a natural involution operator denoted $*$, defined as $\boldsymbol{\alpha}^* = \sum_\alpha \bar{c}_\alpha \, \alpha^{-1}$, for any $\boldsymbol{\alpha} = \sum_\alpha c_\alpha \, \alpha$.

If the groupoid $\mathbf{G}$ is finite, there is a natural unit element $\mathbf{1} = \sum_{a\in\Omega} 1_a$ in the algebra $\mathbb{C}[\mathbf{G}]$. From Eq. (2) we get:

$$\mathbb{C}[\mathbf{G}] \circ 1_a \circ \mathbb{C}[\mathbf{G}] = \mathbb{C}[\mathbf{G}(a)] \, ,$$

with $\mathbb{C}[\mathbf{G}(a)]$ the groupoid algebra of the subgroupoid $\mathbf{G}(a)$.

Another family of relevant mixed transitions are given by $\mathbf{1}_{G_a} = \sum_{\gamma_a \in G_a} \gamma_a$, which are the characteristic 'functions' of the isotropy groups G_a and $\mathbf{1}_{\mathbf{G}_\pm(a)} = \sum_{\alpha\in\mathbf{G}_\pm(a)} \alpha$ that represent the characteristic 'functions' of the sprays $\mathbf{G}_\pm(a)$ at a. Finally, we should mention the 'incidence' or total transition, defined as $\mathbb{I} = \sum_\alpha \alpha$. Clearly,

$$\mathbb{C}[\mathbf{G}] \circ \mathbb{I} = \mathbb{I} \circ \mathbb{C}[\mathbf{G}] = \mathbb{C}[\mathbf{G}] \, ,$$

and

$$\mathbb{I} \circ 1_a = \mathbf{1}_{\mathbf{G}_+(a)} \, , \quad 1_a \circ \mathbb{I} = \mathbf{1}_{\mathbf{G}_-(a)} \, , \quad 1_a \circ \mathbb{I} \circ 1_a = \mathbf{1}_{G_a} \, . \tag{3}$$

3. Amplitudes and Observables

3.1. The algebra of amplitudes. According to the premises laid on in [1] we will assume that the description of a given physical system may be given in terms of groupoids. Specifically, we start with a family $\mathscr{A}$ of experimental setups by means of which we may perform experiments on the physical system under investigation in order to measure a 'property'. The outcomes of measurements performed in such experiments are the 'physical events', and the set of all such outcomes is denoted by $\Omega_{\mathscr{A}}$. According to [1], we will not try to make precise at this stage the meaning of 'measurement', 'property' or the nature of the outcomes as we will consider them primary notions determined solely by the experimental setting used to study our system.

In the incipit of [2], Schwinger writes: "The classical theory of measurement is implicitly based upon the concept of an interaction between the system of interest and the measuring apparatus that can be made arbitrarily small, or at least precisely compensated, so that one can speak meaningfully of an idealized experiment that disturbs no property of the system. The classical representation of physical quantities by numbers is the identification of all properties with the results of such nondisturbing measurements. It is characteristic of atomic phenomena, however, that the interaction between system and instrument cannot be indefinitely weakened. Nor can the disturbance produced by the interaction be compensated precisely since it is only statistically predictable. Accordingly, a measurement on one property can produce unavoidable changes in the value previously assigned to another property, and it is without meaning to ascribe numerical values to all the attributes of a microscopic system. The mathematical language that is appropriate to the atomic domain is found in the *symbolic transcription of the laws of microscopic measurement*[3]".

The "ontological disturbance" of the act of measuring individuated by Schwinger is at the roots of the introduction of the notion of *transitions among the outcomes of experiments*. In a purely classical context, the act of measuring does not influence the system and we may safely say that, if the outcome of the measurement we actually performed on the system is a, the measured property of the system has the value a. On the other hand, this is no longer the case for microscopic phenomena where the outcome a of the measurement of some property we actually performed on the system is compatible with different values, say, a', a'', etc., of the same property before the act of measurement. The *transitions among the outcomes of experiments* (henceforth simply: *transitions*) are precisely the objects that take this instance into account. By imposing a small set of "natural" axioms on it, the set $\mathbf{G}_{\mathscr{A}}$ of transitions becomes a groupoid over the set $\Omega_{\mathscr{A}}$ of events.

An amplitude of the system is by definition a map $f\colon \mathbf{G}_{\mathscr{A}} \to \mathbb{C}$, that is, an assignement of a complex number $f(\alpha)$ to any transition α. The set $\mathcal{F}(\mathbf{G}_{\mathscr{A}})$ of all amplitudes is an algebra with respect to the convolution product:

$$(f \star g)(\gamma) = \sum_{\alpha\circ\beta=\gamma} f(\alpha)g(\beta)\,. \tag{4}$$

where the summation is taken over all transitions α, β in $\mathbf{G}$ such that $\alpha \circ \beta = \gamma$. Notice that the previous expression can also be written as:

$$(f \star g)(\gamma) = \sum_{t(\alpha)=t(\gamma)} f(\alpha)g(\alpha^{-1}\circ\gamma) = \sum_{s(\beta)=s(\gamma)} f(\gamma\circ\beta^{-1})g(\beta)\,.$$

In general, the algebra $\mathcal{F}(\mathbf{G}_{\mathscr{A}})$ of amplitudes is non-commutative. However, there is a natural involution operator $*\colon \mathcal{F}(\mathbf{G}_{\mathscr{A}}) \to \mathcal{F}(\mathbf{G}_{\mathscr{A}})$, $f \mapsto f^*$, defined by:

$$f^*(\gamma) = \overline{f(\gamma^{-1})}\,,$$

that makes $\mathcal{F}(\mathbf{G}_{\mathscr{A}})$ into a $*$-algebra. The observables are then the real elements of the algebra $\mathcal{F}(\mathbf{G}_{\mathscr{A}})$ with respect to the involution $*$. If the groupoid $\mathbf{G}_{\mathscr{A}}$ is discrete countable (or finite), there is a unit element given by the function $\mathbf{1}$ that takes the value 1 on all unit transitions $1_a\colon a \to a$, and zero otherwise, that is:

[3]The emphasizing is due to the authors.

$\mathbf{1} = \delta_{\Omega_{\mathscr{A}}}$, the characteristic function of the set of events $\Omega_{\mathscr{A}}$ considered as a subset of $\mathbf{G}_{\mathscr{A}}$. Notice that:

$$(\mathbf{1} \star f)(\gamma) = \sum_{\alpha} \mathbf{1}(\alpha^{-1} \circ \gamma) f(\alpha) = f(\gamma)\,,$$

and similarly $f \star \mathbf{1} = f$. Furthermore, there is a natural norm defined on $\mathcal{F}(\mathbf{G}_{\mathscr{A}})$ that makes it into a C^*-algebra[4]. In what follows we will assume that the algebra of amplitudes carries a C^*-algebra structure.

It is easy to see that $\mathcal{F}(\mathbf{G}_{\mathscr{A}})$ is 'dual' to the groupoid algebra $\mathbb{C}[\mathbf{G}_{\mathscr{A}}]$ introduced in section 2. Specifically, any function $f \in \mathcal{F}(\mathbf{G}_{\mathscr{A}})$ can be written as:

$$f = \sum_{\gamma} f(\gamma)\delta_\gamma\,,$$

with δ_γ the function that takes the value 1 at γ and zero elsewhere. There is a natural pairing $\langle \cdot, \cdot \rangle : \mathcal{F}(\mathbf{G}_{\mathscr{A}}) \times \mathbb{C}[\mathbf{G}_{\mathscr{A}}] \to \mathbb{C}$, between the algebra of amplitudes and the groupoid algebra obtained by extending linearly the evaluation of amplitudes on transitions, that is:

$$\langle f, \boldsymbol{\alpha} \rangle = \sum_{\alpha} f(\alpha) c_\alpha\,,$$

with $\boldsymbol{\alpha} = \sum_\alpha c_\alpha \alpha$. When $\Omega_{\mathscr{A}}$ is discrete, there is also a natural algebraic identification between both algebras provided by the linear basis $\{\delta_\alpha\}$ and $\{\alpha\}$ of the algebras $\mathcal{F}(\mathbf{G}_{\mathscr{A}})$ and $\mathbf{C}[\mathbf{G}_{\mathscr{A}}]$ respectively. Under this identification the unit $\mathbf{1}$ in $\mathbb{C}[\mathbf{G}_{\mathscr{A}}]$ goes into the unit function $\mathbf{1}$ in $\mathcal{F}(\mathbf{G}_{\mathscr{A}})$.

We may describe this identification by denoting by $\boldsymbol{\alpha}_f$ the element in $\mathbb{C}[\mathbf{G}_{\mathscr{A}}]$ associated with the function f and by $f_{\boldsymbol{\alpha}}$ the function associated with $\boldsymbol{\alpha}$. Then, it is immediate to check that:

$$f_{\boldsymbol{\alpha}} \star f_{\boldsymbol{\beta}} = f_{\boldsymbol{\alpha}\cdot\boldsymbol{\beta}}\,, \qquad \boldsymbol{\alpha}_f \cdot \boldsymbol{\alpha}_g = \boldsymbol{\alpha}_{f\star g}\,.$$

Moreover:

$$\boldsymbol{\alpha}_{f^*} = \boldsymbol{\alpha}_f^*\,, \qquad f_{\boldsymbol{\alpha}^*} = f_{\boldsymbol{\alpha}}^*\,.$$

It is then clear that, under suitable conditions of completeness for the norms on $\mathbb{C}[\mathbf{G}_{\mathscr{A}}]$ and $\mathcal{F}(\mathbf{G}_{\mathscr{A}})$, the algebra of amplitudes $\mathcal{F}(\mathbf{G}_{\mathscr{A}})$ has the structure of a von Neumann algebra because it is the dual Banach space of $\mathbb{C}[\mathbf{G}_{\mathscr{A}}]$. This situation agrees with what happens in the algebraic formulation of quantum field theories where the relevant algebras turns out to be von Neumann algebras.

[4]There is a natural way of constructing a C^*-algebra for a given groupoid over a locally compact space of events by means of a family of (left-invariant) Haar measures as described for instance in [19] (see also [20, Part III, Chap. 3] and references therein).

3.2. **Observables and self-adjoint operators in the fundamental representation.** The fundamental representation of the groupoid $\mathbf{G}_{\mathscr{A}}$ provides a natural intepretation of amplitudes in terms of operators. That is, if we denote as in [1] by $\pi\colon \mathcal{F}(\mathbf{G}_{\mathscr{A}}) \to \mathrm{End}(\mathcal{H}_{\mathscr{A}})$ the fundamental representation of the finite groupoid $\mathbf{G}_{\mathscr{A}}$, which is given by[5]:

$$\pi(f)|a\rangle = \sum_{\alpha} f(\alpha)\delta(\alpha, a)|t(\alpha)\rangle\,, \tag{6}$$

where $a \in \Omega_{\mathscr{A}}$, $|a\rangle$ denotes the corresponding vector in $\mathcal{H}_{\mathscr{A}}$, $\delta(\alpha, a)$ is the indicator function defined as $\delta(\alpha, a) = 1$ if $\alpha\colon a \to b$ and zero otherwise, and $t(\alpha)$ is the target of α, i.e., $t(\alpha) = b$, then:

$$\pi(f^*) = \pi(f)^\dagger\,,$$

that is, the fundamental representation is a $*$-representation. Using an alternative notation $A_f = \pi(f)$, we get $A_{f^*} = A_f^\dagger$, where $A^\dagger$ denotes the adjoint operator of A in $\mathcal{H}_{\mathscr{A}}$.

Notice that if the space of events is finite, a ranges over a finite set and $\mathcal{H}_{\mathscr{A}}$ is a finite dimensional Hilbert space. Notice that $\langle b, A_f a\rangle$ is just the sum of the values of the function f on the transitions $\alpha\colon a \to b$, that is:

$$\langle b, A_f a\rangle = \langle b|(A_f|a\rangle) = \sum_{\alpha\colon a\to b} f(\alpha)\,.$$

where we are using Dirac's notation $\langle b|a\rangle$ to denote the inner product of the vectors $|a\rangle$ and $|b\rangle$. Notice finally that real elements in the algebra $\mathcal{F}(\mathbf{G}_{\mathscr{A}})$, that is, functions such that $f^* = f$, are such that $A_f = A_f^\dagger$. In other words, real elements in the algebra of amplitudes determine self-adjoint operators on the Hilbert space $\mathcal{H}_{\mathscr{A}}$, that is, observables in the standard framework of quantum mechanics. Accordingly, we call a real element in $\mathcal{F}(\mathbf{G}_{\mathscr{A}})$ an *observable*.

For any amplitude f we may write the following formula for the sum of amplitudes:

$$\langle a|A_f|b\rangle = \sum_{\alpha\colon a\to b} f(\alpha)\,.$$

In the particular instance when f is an observable and $a = b$, we get the real number $\langle a|A_f|a\rangle$, that can be interpreted as the expected value of the observable

[5]There is a natural extension of this formula when the groupoid $\mathbf{G}_{\mathscr{A}}$ is a locally compact groupoid over a standard Borel measurable space with a measure μ and a family of left-invariant Haar measures ν_a. In such case $\mathcal{H}_{\mathscr{A}} = L^2(\Omega, \mu)$ and next equation (6) becomes:

$$\pi(f)|a\rangle = \int_{s^{-1}(a)} f(\alpha)\,|t(\alpha)\rangle\, d\nu(\alpha)\,. \tag{5}$$

f in the 'state' $|a\rangle$, given by:

$$\langle a|A_f|a\rangle = \sum_{\alpha\colon a\to a} f(\alpha) = \sum_{\alpha\in G_a} f(\alpha)\,. \tag{7}$$

This formula justifies the name of amplitudes given before to the values of the functions f on transitions. Actually, if there is just one transition from a to b like in Schwinger's measurement algebra model (see [1]), then the value $f(\alpha)$ is exactly the amplitude of the operator $\pi(f) = A_f$ with respect to the vectors $|a\rangle$ and $|b\rangle$ in the fundamental Hilbert space $\mathcal{H}_{\mathscr{A}}$.

3.3. **Completeness of systems of compatible observables.** Notice that the notion of observable we have introduced is consistent with the terminology introduced from the very beginning where the events a were named after the outcomes of measurements performed during some experiment on the system. In fact, given an event a, if we assume for simplicity that a is just a real number, there is an observable in $\mathcal{F}(\mathbf{G}_{\mathscr{A}})$ whose expected value is a. Indeed, the observable $f_a = a\,\delta_{1_a}$ is such that $\langle a|A_{f_a}|a\rangle = a$.

So far, we have identified the algebra of amplitudes associated with the family $\mathscr{A}$ of experimental setups with the dual algebra of the algebra of the groupoid of transitions $\mathbf{G}_{\mathscr{A}}$. No assumption whatsoever was made on the structure of the whole family of amplitudes themselves $\mathcal{A}$. It is possible that when we use a family $\mathscr{A}$ of compatible experimental setups, the algebra of amplitudes $\mathcal{F}(\mathbf{G}_{\mathscr{A}})$ associated with the groupoid of transitions over the space of events $\Omega_{\mathscr{A}}$ determined by $\mathscr{A}$, yield all amplitudes of the system.

More formally, suppose that $\mathcal{A}$ is the family of all amplitudes of the system[6]. Then, we proceed to determine experimentally as many families of events and transitions among them as possible by selecting families of compatible experimental setups $\mathscr{A}$, $\mathscr{B}$, etc. As it was discussed in [1], these families form a groupoid $\mathbf{G}$ with total space of objects Ω. Suppose that we select a family $\mathscr{A}$ of experimental setups and its corresponding subspace of events $\{a\} = \Omega_{\mathscr{A}} \subset \Omega$. This choice will select a subgroupoid $\mathbf{G}_{\mathscr{A}} \subset \mathbf{G}$ consisting of those transitions $\alpha\colon a \to a'$, $a, a' \in \Omega_{\mathscr{A}}$. Eventually, we can consider the algebra of the groupoid $\mathbf{G}_{\mathscr{A}}$ and its corresponding algebra of amplitudes $\mathcal{F}(\mathbf{G}_{\mathscr{A}})$. This algebra will be contained in $\mathcal{A}$ as it was shown before. It could also happen that the groupoid of transitions associated with the family $\mathscr{A}$ of experimental setups we have chosen is 'generic' enough, so that the

[6]Notice that this is just an idealisation of a situation that would never happen, that is, we could never know for sure if the quantities we have identified as measurable for a given system are all its physical attributes that can be measured. For instance, think to the spin of the electron. When Thompson identified it, it was just possible to measure its position, linear momentum, angular momentum, energy and charge. Only much later it was realised that there was another measurable physical quantity for the electron, its spin. We may also consider the examples provided by the many quantum charges, isospin, barionic charge, strangeness, etc., that have been discovered discovered later on and which are characteristic measurable quantities of elementary particles.

algebra of amplitudes $\mathcal{F}(\mathbf{G}_{\mathscr{A}})$ is essentially[7] the whole $\mathcal{A}$. Then we will say that the family of amplitudes associated with $\mathscr{A}$ is a complete[8] family of amplitudes for $\mathcal{A}$. As we were saying before, that an algebra of amplitudes is complete or not could be more an academic question than a real one, in the sense that if we find a family such that the C^*-algebra of amplitudes constructed from them contains all other relevant descriptions we have of the system, we may consider that algebra is just the algebra of amplitude of the system.

In what follows we will just assume that we have a family $\mathscr{A}$ of experimental setups such that the algebra of amplitudes of the system is given by the algebra $\mathcal{F}(\mathbf{G})$ functions on the groupoid $\mathbf{G}$ defined by such family. This is not really a simplifying assumption, as the structure of the events determined by that family could be very complicated. We will often use the simplifying assumption that the space of events is discrete (or even finite) to illustrate the main ideas without having to rely on heavy technical machinery from functional analysis and operator algebras.

4. States

We can now discuss properly the notion of states for physical systems described by groupoids of transitions. Given that the algebra of amplitudes of the system under consideration is a C^*-algebra, the C^*-algebra of functions $\mathcal{F}(\mathbf{G})$ on the groupoid $\mathbf{G}$ of transitions, we define a state ρ as a state on $\mathcal{F}(\mathbf{G})$ in the sense of functional analysis. Consequently, a state ρ is a normalized positive linear functional on $\mathcal{F}(\mathbf{G})$, that is, $\rho\colon \mathcal{F}(\mathbf{G}) \to \mathbb{C}$, is a linear map such that $\rho(f^* \star f) \geq 0$, for all f, and $\rho(\mathbf{1}) = 1$. Notice that we are assuming that the C^*-algebra $\mathcal{F}(\mathbf{G})$ is unital.

According to the previous definition a state is an element in the dual space of $\mathcal{F}(\mathbf{G})$, however, $\mathcal{F}(\mathbf{G})$ is the dual of the groupoid algebra $\mathbb{C}[\mathbf{G}]$ generated by transitions, and thus we may identify some of these transitions as states in the above sense.

For instance consider the linear functional defined by the unit 1_a, that is, $\rho_a(f) = f(1_a)$. Clearly ρ_a is a state because $\rho_a(\mathbf{1}) = \mathbf{1}(1_a) = 1$ and

$$
\begin{aligned}
\rho_a(f^* \star f) &= (f^* \star f)(1_a) = \sum_{\alpha\circ\beta = 1_a} f^*(\alpha) f(\beta) = \sum_{\beta\colon a\to b} f^*(\beta^{-1}) f(\beta) \\
&= \sum_{\beta\colon a \to b} \overline{f(\beta)} f(\beta) = \sum_{\beta\colon a\to b} |f(\beta)|^2 \geq 0
\end{aligned} \tag{8}
$$

where the sum above should be replaced by an integral in the continuous case. Thus the events a obtained from the family $\mathscr{A}$ can be properly identified with

[7] In the infinite dimensional situation we will demand that the algebra of amplitudes generated by $\mathbf{G}_{\mathscr{A}}$ will be dense in $\mathcal{A}$ using an appropriate topology.

[8] Notice that this is not the usual meaning of 'complete' that usually refers to the family to be a maximal subset of compatible observables.

states ρ_a of the algebra of amplitudes. Even more, the value $\rho_a(f) = f(1_a)$ is just the expected value of the amplitude f in the state ρ_a in agreement with the interpretation provided by formula (7) in the case that there is a unique transition $1_a\colon a \to a$. Notice that if the system has 'inner' structure, that is $G_a \neq \{1_a\}$, then the state describing the expected value of the amplitude f would be the state defined as:

$$\rho_a^{\rm inner}(f) = \frac{1}{|G_a|} \sum_{\alpha \in G_a} f(\alpha)\,,$$

or, equivalently $\rho_a^{\rm inner} = \frac{1}{|G_a|}\sum_{\alpha\in G_a}\alpha$, which is a convex combination with weights $p_\alpha = 1/|G_a|$ of all 'inner' transitions $\alpha \in G_a$.

Given a state ρ, we can construct the GNS Hilbert space $\mathcal{H}_\rho$ associated with it and the corresponding representation of the C^*-algebra. Let us recall that $\mathcal{H}_\rho$ is the completion of the quotient space $\mathcal{F}(\mathbf{G})$ with respect to the Gelfand ideal $\mathcal{J}_\rho = \{f \mid \rho(f^*\star f) = 0\}$. There is a natural inner product defined on $\mathcal{F}(\mathbf{G})/\mathcal{J}_\rho$ given by $\langle f + \mathcal{J}_\rho, g + \mathcal{J}_\rho\rangle = \rho(f^*\star g)$ whose associated norm is used to construct the desired completion. The algebra $\mathcal{F}(\mathbf{G})$ is represented canonically on $\mathcal{H}_\rho$ as: $\pi_\rho(f)(g + \mathcal{J}_\rho) = f\star g + \mathcal{J}_\rho$.

In the particular instance when we use the state ρ_a, we get that because Eq. (8), $\rho_a(f^*\star f) = 0$ iff $\sum_\beta |f(\beta)|^2 = 0$, for all $\beta\colon a \to a'$. We will denote by $\mathbf{G}(a)$ the collection of transitions starting at a:

$$\mathbf{G}(a) = \{\alpha\colon a \to a'\}\,,$$

then, the ideal $\mathcal{J}_{\rho_a} = \{f \mid f(\beta) = 0, \beta\colon a \to b\}$, is just the ideal of functions vanishing at $\mathbf{G}(a)$, but then:

$$\mathcal{F}(\mathbf{G})/\mathcal{J}_{\rho_a} = \mathcal{F}(\mathbf{G}(a))\,.$$

Thus, the Hilbert space $\mathcal{H}_{\rho_a}$ of the GNS representation of the state ρ_a is given by the set of functions ψ on $\mathbf{G}(a)$ with inner product:

$$\langle\phi,\psi\rangle_{\rho_a} = \rho_a(\phi\star\psi) = (\phi\star\psi)(1_a) = \sum_{\alpha\in\mathbf{G}(a)} \overline{\phi(\alpha^{-1})}\psi(\alpha)\,, \tag{9}$$

were, with an evident abuse of notation, we use the symbols ϕ and ψ for both the functions in $\mathcal{F}(\mathbf{G}(a))$ and their extension to $\mathcal{F}(\mathbf{G})$.

Finally, notice that the space $\mathcal{H}_{\rho_a} = \mathcal{F}(\mathbf{G}(a))$ supports the GNS representation of the algebra $\mathcal{F}(\mathbf{G})$, that is, $\mathcal{F}(\mathbf{G})$ acts on it by $\pi_a(f)\psi = f\star\psi$. This action is the dual action of the action of the groupoid algebra $\mathbb{C}[\mathbf{G}]$ in $\mathbf{G}(a)$ on the right, that is: $\alpha\mapsto\alpha\circ\beta$, for all $\alpha\colon a\to a'$ and β a transition composable with α[9]. We have concluded the GNS construction for the state ρ_a

[9]The 'duality' between $\mathbb{C}[\mathbf{G}]$ and $\mathcal{F}(\mathbf{G})$ must be defined properly (that is being antilinear in the first factor) so that everything fits nicely - recall the problem with the adjoint in the fundamental representation!

On the other hand, notice that the isotropy group G_a of the unit 1_a is contained in $\mathbf{G}(a)$ and it acts on $\mathbf{G}(a)$ by composition on the right, that is, $\gamma_a \colon \alpha \to \gamma_a \circ \alpha$, $\gamma_a \in G_a$ and $\alpha \in \mathbf{G}(a)$. Then, provided the groupoid $\mathbf{G}$ is connected, we have :

$$\mathbf{G}(a)/G_a \cong \Omega \, .$$

The quotient space $\mathbf{G}(a)/G_a$ (that is, the space of orbits of G_a in $\mathbf{G}(a)$) is in one-to-one correspondence with the space of events Ω. The map describing such correspondence is given by $[\alpha] \mapsto t(\alpha) = a'$ if $\alpha \colon a \to a'$, and $[\alpha]$ denotes the orbit passing through α. The map is clearly surjective. To show that it is injective, notice that $t(\gamma_a \circ \alpha) = t(\alpha)$ and if we have two transitions: $\alpha, \alpha' \colon a \to a'$, then $\alpha' \circ \alpha^{-1} = \gamma_a \in G_a$ and $[\alpha] = [\alpha']$.

The GNS representation π_a will not be irreducible in general, that is, the state ρ_a is not pure in general. We can see that by observing that there is a natural representation μ_a of the group G_a on $\mathcal{H}_{\rho_a} = \mathcal{F}(\mathbf{G}(a))$ defined as follows:

$$[\mu_a(\gamma_a)\psi](\alpha) = \psi(\gamma_a \circ \alpha) \, , \qquad \gamma_a \in G_a, \alpha \in \mathbf{G}(a) \, ,$$

and $\psi \colon \mathbf{G}(a) \to \mathbb{C}$ is a function in $\mathcal{H}_{\rho_a}$. Notice that the representation μ_a will not be irreducible in general and it will decompose as a direct sum of irreducible representations of G_a. However μ_a will always contain the trivial representation of G_a. It will be given by the subspace of invariant functions in $\mathcal{F}(\mathbf{G}(a))$, that is, the subspace of functions of the form:

$$\tilde{\psi}(\alpha) = \frac{1}{\sqrt{|G_a|}} \sum_{\gamma_a \in G_a} \psi(\gamma_a \circ \alpha) \, .$$

Notice that this subspace, that can be denoted as $\tilde{\mathcal{H}}_\Omega$, is isomorphic to the Hilbert space $\mathcal{H}_\Omega$ supporting the fundamental representation because these functions are invariant along the orbits of G_a, so that they project to functions on $\mathbf{G}(a)/G_a \cong \Omega$. The precise assignment is given by $\tilde{\psi} \mapsto \psi$, $\psi(a') = \tilde{\psi}(\alpha)$, with $\alpha \colon a \to a'$.

Finally, notice that because of Eq. (9) we get:

$$\langle \tilde{\phi}, \tilde{\psi} \rangle_{\rho_a} = \sum_{\alpha \in \mathbf{G}(a)} \overline{\tilde{\phi}(\alpha^{-1})} \tilde{\psi}(\alpha) = \frac{1}{|G_a|} \sum_{a' \in \Omega} \sum_{\gamma_a \in G_a} \overline{\phi}(a') \psi(a') = \langle \phi, \psi \rangle_{\mathcal{H}_\Omega} \, ,$$

which shows that the trivial irreducible component $\tilde{H}_{\rho_a}$ of the GNS representation $\mathcal{H}_{\rho_a}$ of the state ρ_a is isomorphic to the fundamental representation of the algebra of observables of the groupoid. $\mathbf{G}$. Evenually, we can summarise the results obtained is far in the following theorem:

Theorem 1. *Given a physical system described by the groupoid $\mathbf{G}$ of transitions among the outcomes of experiments associated with a family $\mathscr{A}$ of experimental setups with outcome space Ω, and such that the algebra $\mathcal{F}(\mathbf{G})$ of amplitudes of the system is a C^*-algebra with unit, there is a Hilbert space associated with the system which is provided by the Hilbert space $\mathcal{H}_\Omega$ supporting the fundamental representation of the groupoid $\mathbf{G} \rightrightarrows \Omega$. Moreover, the states ρ_a determined by the unit* *transitions* 1_a *of the system are naturally identified with the vectors* $|a\rangle \in \mathcal{H}_\Omega$. *The Hilbert space* $\mathcal{H}_\Omega$ *is isomorphic to the subspace supporting the trivial representation of the group* G_a *in the Hilbert space* $\mathcal{H}_{\rho_a}$ *obtained by the GNS construction applied to any state* ρ_a. *Eventually, observables in the algebra* $\mathcal{F}(\mathbf{G_A})$ *are represented as self-adjoint operators in* $\mathcal{H}_\Omega$. *The expected value of the real observable* f *in any state* ρ *determining a vector* $|\phi\rangle$ *in* $\mathcal{H}_\Omega$ *is given by* $\langle f \rangle_\rho = \rho(f) = \langle \phi | A_f | \phi \rangle$.

5. Schwinger's transition functions

5.1. A 'relativity principle' and the composition of transitions again.

The assumption that we can construct the algebra of observables of the system out of a complete family of compatible experimental setups and its corresponding groupoid of transitions leads to a relevant observation regarding the nature and composition properties of transitions.

Two complete families of experimental setups $\mathscr{A}$, $\mathscr{B}$ for a given physical system provide two different descriptions of its family of observables $\mathcal{A}$ given respectively by the algebras $\mathcal{F}(\mathbf{G}_{\mathscr{A}})$ and $\mathcal{F}(\mathbf{G}_{\mathscr{B}})$. Because the physical reality described by observers using an experimental setting $\mathscr{A}$ cannot be different from that described by other observers using $\mathscr{B}$, we postulate that the algebras $\mathcal{F}(\mathbf{G}_{\mathscr{A}})$ and $\mathcal{F}(\mathbf{G}_{\mathscr{B}})$ must be isomorphic. Given their canonical C^*-algebraic structures, we will assume that they are isomophic as C^*-algebras. In fact, this assumption is based on physical grounds as the involution operator $*$ is the abstract notion of the adjoint operator in the fundamental representation, thus the condition that the identification between both algebras is a $\star$-homomorphism is just the demand that the identification preserves the identification of real observables. On the other hand, the norms of the algebras are induced from the fundamental representation, thus the condition the the identification is norm preserving is just the statement that the identification of amplitudes $f(\alpha)$ with expectation values (recall Eq. (7)) is preserved.

This equivalence between the physical realities described by using different complete families of experimental setups is a sort of relativity principle that has deep implications on the composition properties of transitions. In fact, if $\mathscr{A}$ and $\mathscr{B}$ represent again two complete descriptions of the system, the algebras generated by the transitions of both systems, that is the algebras of the corresponding groupoids $\mathbf{G}_{\mathscr{A}}$ and $\mathbf{G}_{\mathscr{B}}$ to which the corresponding algebras of observables are dual, must be isomorphic too because of the equivalence of the algebras of observables. We can denote by $\tau\colon \mathbb{C}[\mathbf{G}_{\mathscr{B}}] \to \mathbb{C}[\mathbf{G}_{\mathscr{A}}]$ this isomorphim and by $\tau^*\colon \mathcal{F}(\mathbf{G}_{\mathscr{A}}) \to \mathcal{F}(\mathbf{G}_{\mathscr{B}})$ the corresponding isomorphism between the algebras of observables.

Notice that the transitions are observed experimentally and they occur independently of the devices we have chosen to set our experimental setting. However, the composition law on each groupoid $\mathbf{G}_{\mathscr{A}}$ depends on the events determined by $\mathscr{A}$, hence, the groupoid algebra law depends on the chosen system $\mathscr{A}$. This implies that when observing a transition $\beta\colon b \to b'$ within the 'experimental frame' provided by the system $\mathscr{A}$ we do not get a yes-no answer as it would be the case when observing a transition $\alpha\colon a \to a'$ with events a, a' defined by $\mathscr{A}$. However, because of the isomorphism τ between both representation it is possible to identify the transition β with an element in the algebra $\mathbb{C}[\mathbf{G}_{\mathscr{A}}]$, that is:

$$\tau(\beta) = \sum_{\alpha \in \mathbf{G}_{\mathscr{A}}} c(\beta, \alpha)\alpha \,, \tag{10}$$

for some complex numbers $c(\beta, \alpha)$[10].

This decomposition of transitions β corresponding to a given 'experimental frame' $\mathscr{B}$ with respect to transitions in a different, hence necessarily incompatible, experimental frame $\mathscr{A}$ is instrumental in Schwinger's construction of the algebra of measurements. Let us recall (see [1]) that 'transitions' are realised in Schwinger's algebra of measurements by means of selective measurements $M_A(a, a')$, meaning by that a device that selects the system whose outcome when measuring $\mathscr{A}$ is a and returns the system changed in such a way that the outcome of another measure of $\mathscr{A}$ would be a'. Thus, in principle, it does not make sense to compose selective measurements $M_A(a, a')$ and $M_B(b, b')$ corresponding to incompatible systems of experimental setups (unless the events a' and b are equivalent, as it was observed in [1]). However, at this point, in order to develop a full algebra of measurements, Schwinger introduces the following fundamental assumption [3, pp. 9]:

"...(selective) Measurements that we have already considered involve the passage of all systems or no systems at all between the two stages, as represented by the multiplicative numbers 1 and 0. More generally, measurements of properties $\mathscr{B}$, performed on a system in a state a' that refers to properties incompatible with $\mathscr{B}$, will yield a *statistical distribution*[11] of possible values. Hence only a determinate *fraction* of the systems emerging from the first state will be accepted by the second stage. We express this by the general multiplication law:

$$M(a', b')M(c', d') = \langle b' \mid c' \rangle M(a', d') \,, \tag{11}$$

where $\langle b' \mid c' \rangle$ is a number characterizing the statistical relation between the states b' and c'."

Even if at first sight this interpretation of the experimental results seems to be correct, there is a fundamental issue with it. A proper probabilistic interpretation of the fraction of the systems that will emerge in the final state should be given by a positive real number, while the numbers $\langle b' \mid c' \rangle$ appearing in the previous expansion are complex and as such are treated in Schwinger's construction of the algebra of measurements (see for instance, Eq. (1.40) in [3, pp. 16]). Actually they must be so because they represent amplitudes of transitions. It is the positive

[10] Properly speaking, it would be the image of β under the isomorphism between the two algebras the one that would be written as a linear combination of transitions in $\mathbf{G}_{\mathscr{A}}$, but in what follows we will identify β with its image to avoid cumbersome notations.

[11] The underlying is ours.

real number $|\langle b' \mid c'\rangle|^2$ the one that provides the probabilistic interpretation and the one that is actually measured in experiments.

Thus, we conclude that Schwinger's interpretation of the composition law for compound measurements Eq. (11) should be properly interpreted. A proper interpretation is provided by formula (10) above. To be more precise, the fundamental property that we have established is that a given physical transition β can be described as a linear combination with complex coefficients of transitions $\alpha\colon a \to a'$ obtained from a different complete family of experimental setups $\mathscr{A}$. Hence given two transitions $\alpha\colon a \to a' \in \mathbf{G}_{\mathscr{A}}$ and $\beta\colon b \to b' \in \mathbf{G}_{\mathscr{B}}$, we can compose them once we identify β with an element in $\mathbb{C}[\mathbf{G}_{\mathscr{A}}]$ (or viceversa).

5.2. **General Schwinger's transition functions.** Even if the composition formula (10) provides a way to interpret the experimental results obtained when observing transitions between events with respect to different complete systems of observables, we have not provided a way of describing the general class of compound transitions $M(a,b)$ used in Schwinger's composition law, Eq. (11), and that we have called Stern-Gerlach transitions in [1].

We will start by defining a compound (or generalised, or Stern-Gerlach) transition $\gamma_{ab}\colon a \to b \in \mathbb{C}[\mathbf{G}_{\mathscr{A}}]$ as follows:

$$\gamma_{ab} := \sum_{\alpha\colon a\to a'} \alpha \circ \tau(1_b) = \sum_{\alpha\in\mathbf{G}(a)} \alpha \circ \tau(1_b)\,. \tag{12}$$

First we must point out that the transition γ_{ab} lies in the algebra of transitions with respect to the complete system $\mathscr{A}$ and, even if will have a definite outcome b with respect to the system $\mathscr{B}$, it will not have a definite outcome with respect to the system $\mathscr{A}$. In particular, the image $\tau(1_b)$ of the unit transition 1_b (corresponding to the event b defined by the complete system $\mathscr{B}$) will be a linear combination of transitions $\alpha \in \mathbf{G}_{\mathscr{A}}$. However, because the identification τ between the corresponding algebras is an isomorphism of $*$-algebras, we have that $\tau(1_b)^2 = \tau(1_b^2) = \tau(1_b)$ and $\tau(1_b)^* = \tau(1_b^*) = \tau(1_b)$, hence $\tau(1_b)$ is a real idempotent element. In this sense definition (12) of a compound transition can be understood as 'the projection onto the event b of all transitions emanating from a'.

An important observation is that because the algebras $\mathbb{C}[\mathbf{G}_{\mathscr{A}}]$ and $\mathbb{C}[\mathbf{G}_{\mathscr{B}}]$ are isomorphic as C^*-algebras, their corresponding irreducible representations must be unitarily equivalent. Thus the fundamental representation π_A of $\mathbb{C}[\mathbf{G}_{\mathscr{A}}]$ and the fundamental representation π_B of $\mathbb{C}[\mathbf{G}_{\mathscr{B}}]$ are unitarily equivalent. This means that there must exists an unitary operator $U\colon \mathcal{H}_{\mathscr{A}} \to \mathcal{H}_{\mathscr{B}}$ such that we get the covariance property:

$$\pi_A(\tau(\beta)) = U^\dagger \pi_B(\beta) U\,, \qquad \forall \beta \in \mathbf{G}_{\mathscr{B}}\,.$$

Then, recall that the unit transition 1_b defines the vector $|b\rangle \in \mathcal{H}_{\mathscr{B}}$, so that $|b\rangle$ can be identified with a vector $|\tilde{b}\rangle = U^\dagger |b\rangle \in \mathcal{H}_{\mathscr{A}}$ in the Hilbert space supporting the fundamental representation of $\mathcal{F}(\mathbf{G}_{\mathscr{A}})$. In what follows we will use the same symbol for the vector $|b\rangle$ and its image $|\tilde{b}\rangle$ in the space $\mathcal{H}_{\mathscr{A}}$, thus we may write:

$$|b\rangle = \sum_{a\in\Omega_{\mathscr{A}}} \langle a|b\rangle |a\rangle \, .$$

with the complex numbers $\langle a|b\rangle$ denoting the inner product of the vectors $|a\rangle$ and $|\tilde{b}\rangle$ in $\mathcal{H}_{\mathscr{A}}$. Alternatively we may have used that the unit 1_b determines the state ρ_b (that is, $\rho_b(f) = f(1_b)$) on the algebra of observables $\mathcal{F}(\mathbf{G}_{\mathscr{B}})$, but as both algebras $\mathcal{F}(\mathbf{G}_{\mathscr{B}})$ and $\mathcal{F}(\mathbf{G}_{\mathscr{A}})$ are isomorphic, then ρ_b will also define a state in $\mathcal{F}(\mathbf{G}_{\mathscr{A}})$. More precisely, $(\tau^{-1})\rho_b$ will define a state on $\mathcal{F}(\mathbf{G}_{\mathscr{A}})$. Not only that, as it was shown before the state ρ_b can be identified with the vector $|b\rangle$ in the fundamental representation of $\mathcal{F}(\mathbf{G}_{\mathscr{B}})$, but, again because both representations are equivalent, the vector $|b\rangle$ can be identified with a vector in the fundamental representation of $\mathcal{F}(\mathbf{G}_{\mathscr{A}})$.

6. Dynamics

6.1. **A first approach to dynamics on Schwinger's groupoids: Heisenberg representation.** A dynamical description of a physical system consists in prescribing the evolution of its states. In our current setting (see theorem 1), states are positive normalized linear functionals ρ on the algebra $\mathcal{A}$ generated by the observables of the system, where the algebra $\mathcal{A}$ is identified with the C^*-algebra $\mathcal{F}(\mathbf{G})$ with $\mathbf{G}$ the groupoid of transitions of the system. The family of states will be denoted as $\mathcal{S}(\mathbf{G})$ and is a convex set in the topological dual of $\mathcal{A}$.

However, because of the natural duality between states and observables, instead of describing the evolution of states, we may also describe the dynamical evolution of a system by means of observables. In particular, we will consider all those dynamical evolutions that are described as a one-parameter family of positive, normalised linear maps of the C^*-algebra $\mathcal{F}(\mathbf{G})$. Actually, a positive, normalised linear map $\Phi\colon \mathcal{F}(\mathbf{G}) \to \mathcal{F}(\mathbf{G})$, induces a map $\Phi^*\colon \mathcal{S}(\mathbf{G}) \to \mathcal{S}(\mathbf{G})$, as:

$$\Phi^*(\rho)(f) = \rho(\Phi(f)) \, , \qquad \rho \in \mathcal{S}(\mathbf{G}) \, , f \in \mathcal{F}(\mathbf{G}) \, .$$

This approach is the analog of Heisenberg's picture in the setting we are developing. A linear map $\Phi\colon \mathcal{A} \to \mathcal{B}$ is positive if it maps the positive cone of the C^*-algebra $\mathcal{A}$ into the positive cone of the C^*-algebra $\mathcal{B}$. Then, if Φ is positive, Φ^* maps positive linear functionals into positive linear functionals. Finally, if Φ is normalised, that is $\Phi(\mathbf{1}) = \mathbf{1}$, it maps normalised linear functionals into normalised linear functionals, $\Phi^*(\rho)(\mathbf{1}) = \rho(\Phi(\mathbf{1})) = \rho(\mathbf{1}) = 1$. Hence, if Φ is a normalised positive linear map of the C^*-algebra $\mathcal{F}(\mathbf{G})$, then Φ^* maps the state ρ into another state $\Phi(\rho)$ of the system. Consequently, if Φ_t is a one-parameter family of normalised positive maps, the maps $\varphi_t := \Phi_t^*\colon \mathcal{S}(\mathbf{G}) \to \mathcal{S}(\mathbf{G})$ define a dynamical evolution on the space of states.

We will not discuss here the characterisation of positive linear maps[12] and we will leave this discussion for later analysis. What we want to focus our attention on is on the simplest situation of dynamics of closed systems.

A closed system is a system for which its dynamical evolution is independent of external observations. 'Observations' here refers to the collection of actions undertaken by specific observers when preparing and analysing the system. Of course, when measurements are performed, the states of the system can be modified and consequently the subsequent evolution of the states changes, however, no further modifications on the dynamical behaviour of the system are caused by the observers. From the mathematical point of view, this means that the algebra of transitions and their transformations is not affected by the dynamics. In turn, this means that the linear maps Φ_t describing their dynamics must preserve the composition of transitions, hence, they must preserve the convolution product in $\mathcal{F}(\mathbf{G})$:

$$\Phi_t(f \star g) = \Phi_t(f) \star \Phi_t(g)\,.$$

More generally, we may consider that evolution is described by a family $\Phi_{t_0,t}$ of linear transformations of the algebra $\mathcal{F}(\mathbf{G})$, where t_0 indicates a reference time chosen by the observer and $t > t_0$ the time when the system is observed. However, because the system is closed, its dynamical behaviour does not depend on the particular reference t_0 chosen by the observer, and we conclude that $\Phi_{t_0,t}$ depends only on the difference $s = t - t_0$ and $\Phi_{t_0,t} = \Phi_{t-t_0}$. The family of maps Φ_t will be called the dynamical flow of the system.

On the other hand, the system is 'reversible' because it is closed, that is, the knowledge of the evolved states $\rho_t = \Phi^*_{t-t_0}\rho_{t_0}$ at time $t > t_0$ under the dynamic flow Φ_{t-t_0} allows to determine the original states ρ_{t_0} by inverting the dynamics, that is, $\rho_{t_0} = (\Phi^{-1}_{t-t_0})^*\rho_t$. Hence, the dynamical flow should consists of invertible linear maps that, in addition, must satisfy:

$$\Phi_t \circ \Phi_s = \Phi_{t+s}\,.$$

The dynamics is thus described by a one-parameter group of linear invertible maps[13].

Moreover, it is natural to request that the dynamics should preserve the real character of observables, that is, if $f^* = f$, then $\Phi_t(f)^* = \Phi_t(f) = \Phi_t(f^*)$. Consequently, because we may write any element $f \in \mathcal{F}(\mathbf{G})$ as $f = f_1 + if_2$ with f_a real, Φ_t preserves the real character of observables iff $\Phi_t(f)^* = \Phi_t(f)^*$ for all f and all

[12] More precisely, we would like to consider completely positive maps, but this will be discussed elsewhere where the specific adaptation of Stinespring's and Choi's theorems to the C^*-algebra $\mathcal{F}(\mathbf{G})$ will be analysed.

[13] In general, it is only a local one-parameter group of automorphisms as it is not guaranteed that Φ_t is defined for all t.

t. Therefore, we conclude that the dynamical flow Φ_t of a closed system should consists of a one-parameter group of automorphisms of the C^*-algebra $\mathcal{F}(\mathbf{G})$ [14].

Notice that, if the algebra $\mathcal{F}(\mathbf{G})$ is unital, then necessarily $\Phi(\mathbf{1}) = \mathbf{1}$, and thus Φ is normalised. Moreover, if Φ is an automorphims, we have $\Phi(f^* \star f) = \Phi(f)^* \star \Phi(f) \geq 0$ for any f, and thus Φ is positive. Eventually, we conclude that every such family of automorphisms Φ_t defines a family of normalised positive maps.

If we have a dynamical flow Φ_t on the C^*-algebra $\mathcal{F}(\mathbf{G})$, its infinitesimal generator D defined as:

$$Df = \frac{d}{dt}\Phi_t(f) \mid_{t=0} ,$$

is a derivation D, that is, it is a linear map such that $D(f \star g) = Df \star g + f \star Dg$ for all $f, g \in \mathcal{F}(\mathbf{G})$. Moreover, the derivation D is a $*$-derivation, that is $D(f^*) = (Df)^*$, hence it maps real observables into real observables. It is easy to check that given an arbitrary function $k \in \mathcal{F}(\mathbf{G})$ the operation $D_k f = [f, k] = f \star k - k \star f$ is a derivation, moreover if k is imaginary, that is $k^* = -k$, then it defines a $*$-derivation as:

$$(D_k f)^* = [f, k]^* = k^* \star f^* - f^* \star k^* = f^* \star k - k \star f^* = [f^*, k] = D_k(f^*) .$$

We may assume in what follows that the derivation is bounded (what always be the case in finite dimensions) even if this will not be the case in general (see later Sect. 7.2). Moreover if the algebra $\mathcal{F}(\mathbf{G})$ is semisimple, as it happens in the finite-dimensional case [18], then the derivation D will be inner, this means that there will exist an imaginary element $\tilde{h} = ih$ (h real) such that:

$$D = i[\cdot, h] .$$

We will call the real observable h the Hamiltonian generator of the dynamical flow and it will determine the dynamics of the system.

6.2. **The Hamiltonian formalism.** Suppose that a Hamiltonian h is given, then, we may write down the equation for the dynamics of the system in Heisenberg form as:

$$\frac{d}{dt} f = i[f, h] , \tag{13}$$

meaning that, given an initial observable f_0, a solution of Eq. (13) is a curve $f(t)$ of observables such that $df(t)/dt = i[f(t), h]$. Because the derivation $D_h = [\cdot, h]$ is bounded, that is $h \in \mathcal{F}(\mathbf{G})$, we may build its associated dynamical flow as:

$$\Phi_t f = \exp itD_h f = \sum_{k \geq 0} \frac{(it)^k}{k!} D_h^k(f) ,$$

[14] It is often requested that the flow satisfies a continuity property, typically being strongly continuous with respect to the topology of the C^*-algebra.

and after some simple computations we get that the solution to Eq. (13) with initial value f_0 is given by

$$f(t) = e^{itD_h} f_0 = \Phi_t(f_0) \, .$$

which justifies the opening statement of this paragraph.

We should stress that, because the fundamental representation π is a representation of the algebra $\mathcal{F}(\mathbf{G})$, we have $\pi(f \star g) = \pi(f)\pi(h)$, and then Eq. (13) becomes Heisenberg's evolution equation in the standard formalism of operators in Hilbert space, that is:

$$\frac{d}{dt} A = i[A, H] \, . \tag{14}$$

where $H = A_h = \hat{h} = \pi(h)$ is the self-adjoint operator on $\mathcal{H}_\Omega$ representing the Hamiltonian h, and $A = A_f$ for some f. Notice that any operator A is the image under π of some element f in $\mathcal{F}(\mathbf{G})$[15].

Recall from the discussion on Sect. 4 that the folium of the state ρ_x consists of density operators in the fundamental representation, thus, in particular, equation (14), describes the evolution of density operators ('mixed states'), that is:

$$\frac{d}{dt} \hat{\rho} = i[\hat{\rho}, H] \, . \tag{15}$$

This equation is also known as Landau-von Neumann's evolution equation.

6.3. **The quantum-to-classical transition.** As a direct application of the discussion before we may sketch a description of the transition from a purely quantum description of a dynamical system to a classical one. This constitutes a relevant problem in any dynamical description of quantum systems which has not a general agreement on how to be addressed. There are many proposals and ideas on how to address this problem ([22]) some of them close in spirit to the proposal here. A more detailed discussion of it will be pursued elsewhere.

First of all, we shall make precise what a classical description of a physical system is. If we have a system whose algebra of observables is given by $\mathcal{F}(\mathbf{G})$, it has a natural subalgebra provided by the functions supported on Ω[16], that is the algebra of functions $\mathcal{F}(\Omega)$ that can be considered then as a subalgebra of $\mathcal{F}(\mathbf{G})$. Notice that, if the product does not increase the support $\mathrm{supp}(f), \mathrm{supp}(g) \subset \Omega$, then:

$$f \star g = f \cdot g \, ,$$

with $\cdot$ denoting the commutative pointwise product on functions in $\mathcal{F}(\Omega)$. Notice that the representation $\pi(f)$ of a function f with support in Ω is provided by the multiplication operator by the function, then $||\pi(f)|| = \sup ||f \cdot \Psi|| =$

[15] This is a general fact known as the 'density theorem'.

[16] Recall that Ω can be considered as a subset of $\mathbf{G}$ by using the identification of events a with the units 1_a.

$\sup_{a\in\Omega}|f(a)| = ||f||_\infty$, hence $\mathcal{F}(\Omega)$ inherits the structure of a commutative C^*-algebra over Ω. Thus the commutative subalgebra of functions on Ω provides a good model for the space of observables of a classical system whose configurations are the events in Ω. On the other hand, classical states will correspond to normalized positive functional on $\mathcal{F}(\Omega)$. For instance, if Ω is a compact topological space, then the C^*-algebra $\mathcal{F}(\Omega)$ becomes the C^*-algebra of continuous functions on Ω and the space of states the space of Radon measures on Ω.

In order to understand what kind of dynamics is induced on the classical subalgebra $\mathcal{F}(\Omega)$ from a Hamiltonian dynamics on $\mathcal{F}(\mathbf{G})$ we will assume that the Hamiltonian h_ϵ on $\mathbf{G}$ depends on a small parameter ϵ in such a way that $h_\epsilon \to h_0$ when $\epsilon \to 0$,and h_0 is a classical observable, that is $h_0 \in \mathcal{F}(\Omega)$. We will be more precise on the dependence of h_ϵ in a moment. Notice that if f is a classical observable, $f \in \mathcal{F}(\Omega)$, then, if $\alpha\colon x \to y$ is an allowed transition from x to y, we get:

$$[f, h_\epsilon](\alpha) = (f(y) - f(x))h_\epsilon(\alpha)\,,$$

hence,

$$\begin{aligned}
[f, h_\epsilon] &= \sum_{\alpha\colon x\to y} (f(y) - f(x))h_\epsilon(\alpha)\delta_\alpha \\
&= \sum_{\alpha\colon x\to y} f(y)h_\epsilon(\alpha)\delta_\alpha - \sum_{\alpha\colon x\to y} f(x)h_\epsilon(\alpha)\delta_\alpha \\
&= \sum_{x\in\Omega}\left(\sum_{\alpha\in\mathbf{G}_-(x)} f(x)h_\epsilon(\alpha)\delta_\alpha - \sum_{\alpha\in\mathbf{G}_+(x)} f(x)h_\epsilon(\alpha)\delta_\alpha\right) \\
&= \sum_{x\in\Omega}\left(\sum_{\alpha\in\mathbf{G}_+(x)} f(x)\bar{h}_\epsilon(\alpha^{-1})\delta_{\alpha^{-1}} - \sum_{\alpha\in\mathbf{G}_+(x)} f(x)h_\epsilon(\alpha)\delta_\alpha\right) \\
&= \sum_{x\in\Omega} f(x) \sum_{\alpha\in\mathbf{G}_+(x)} \left(\bar{h}_\epsilon(\alpha^{-1})\delta_{\alpha^{-1}} - h_\epsilon(\alpha)\delta_\alpha\right)\,.
\end{aligned}$$

The quantum-to-classical transition from the quantum system $(\mathcal{F}(\mathbf{G}), h_\epsilon)$, $\epsilon > 0$ to a classical system on $\mathcal{F}(\Omega)$ will be obtained by assuming that as $\epsilon \to 0$, the amplitudes of the transitions $\alpha\colon x \to y$, $x \neq y$, tend to zero and become concentrated at the edges, that is, we will assume that hamiltonian h_ϵ has a power series expansion of the form:

$$h_\epsilon(\alpha) = \epsilon h_{1,\alpha}(x, y) + \epsilon^2 h_{2,\alpha}(x, y) + \cdots\,, \qquad \alpha\colon x \to y\,.$$

On the other hand, the basis functions δ_α, will also have to have a limit when $\epsilon \to 0$ in $\mathcal{F}(\Omega)$. The only natural limit form them is $\delta_y - \delta_x$ if $\alpha\colon x \to y$ or, in other words, we may imagine that there is a deformation $\delta\alpha(\epsilon)$ such that $\delta_\alpha(1) = \delta_\alpha$ and $\delta_\alpha(0) = \delta_y - \delta_x$. For instance, if we represent the transition $\alpha\colon x \to y$ as the oriented interval $[0,1]$, and δ_α is the constant function 1, then $\delta_\alpha(\epsilon)$ could be given by the family of functions shown in the picture. Thus we will assume that:

$$\bar{h}_\epsilon(\alpha^{-1})\delta_{\alpha^{-1}} - h_\epsilon(\alpha)\delta_\alpha = \epsilon\left((\bar{h}_{1,\alpha}(x,y) - h_{1,\alpha}(x,y))(\delta_y - \delta_x)\right) + h.o.t.\,,$$

Then, the dynamical evolution of the system is given by:

$$\begin{aligned} \dot{f} &= i[f, h_\epsilon] = \\ &= i\epsilon \sum_{x\in\Omega} f(x) \sum_{y\in\Omega} \sum_{\alpha\in\mathbf{G}(x,y)} \left(\bar{h}_{1,\alpha}(x,y) - h_{1,\alpha}(x,y)\right)(\delta_y - \delta_x) \\ &= \epsilon \sum_{x,y\in\Omega} f(x)k(x,y)(\delta_y - \delta_x)\,, \end{aligned}$$

with the kernel $k(x,y)$ given by:

$$k(x,y) = -2 \sum_{\alpha\in\mathbf{G}(x,y)} \operatorname{Im} h_{1,\alpha}(x,y)\,,$$

and

$$k(x,y) = -k(y,x)\,.$$

If we consider now a change in the scale of time as $t \mapsto \tau = \epsilon t$, then the equation of motion for the classical observable f becomes:

$$\frac{d}{d\tau} f = \sum_{x,y\in\Omega} (f(x) - f(y))k(x,y)\delta_x\,, \tag{16}$$

or, if Ω is finite and its elements numbered $x_1,\ldots,x_n$ and the values $f(x_i) = f_i$, $k(x_i,x_k) = k_{ij}$, then:

$$\frac{d}{d\tau} f_i = \sum_{j=1}^n \left(k_{ij} f_j - k_{ij} f_i\right)\,,$$

thus we defining the matrix K with entries:

$$K_{ij} = k_{ij} - \sum_{l=1}^n k_{il}\delta_{ij}\,, \tag{17}$$

we have:

$$\frac{d}{d\tau} f = K \cdot f\,,$$

with $\cdot$ denoting the matrix vector product and f denoting the column vector with entries f_i.

In particular notice that if we have a classical state, that is a state of the form $p = \sum_{x\in\Omega} p_x 1_x$, $p_x \geq 0$, $\sum_x p_x = 1$, then, its evolution under a Hamiltonian function h_ϵ becomes:

$$\frac{d}{d\tau} p_i = \sum_{j=1}^n K_{ij} \cdot p_j\,, \tag{18}$$

but because Eq. (17) we get that the matrix K satisfies that $\sum_{i=1}^{n} K_{ij} = 0$, and then

$$\frac{d}{d\tau}\sum_{i=1}^{n} p_i = 0\,,$$

and the total probability is conserved. Finally if h_ϵ is such that $h_\epsilon(\alpha) = i\epsilon\gamma(x,y)$ with $\gamma(x,y) < 0$, then $k(x,y) > 0$ and the classical evolution equation of the system is that of a classical random walk on the space of events Ω.

7. Some simple examples

7.1. **The extended singleton.** Let us start the discussion by considering what is arguably the simplest non-trivial groupoid structure. We call it the extended singleton and is given by the diagram below, see Fig. 1:

$$\begin{tikzcd} + \bullet \arrow[r, bend left, "\alpha"] & \bullet \; - \arrow[l, bend left, "\alpha^{-1}"] \end{tikzcd}$$

Figure 1. The extended singleton.

This diagram will correspond to a physical system described by a complete family of experimental setups $\mathscr{A}$ producing just two outputs, denoted by $+$ and $-$ in the diagram, and with just one transition $\alpha\colon + \to -$ among them. Notice that the groupoid $\mathbf{G}_{\mathscr{A}}$ associated to this diagram has 4 elements $\{1_+, 1_-, \alpha, \alpha^{-1}\}$ and the space of events is just $\Omega_{\mathscr{A}} = \{+,-\}$. The groupoid algebra is a complex vector space of dimension 4 generated by $e_1 = 1_+$, $e_2 = 1_-$, $e_3 = \alpha$ and $e_4 = \alpha^{-1}$ with structure constants given by the relations:

$$\begin{aligned} &e_1^2 = e_1\,, \quad e_2^2 = e_2\,, \quad e_1e_2 = 0\,, \quad e_3e_4 = e_1\,, \\ &e_4e_3 = e_2\,, \quad e_3e_3 = e_4e_4 = 0\,, \quad e_1e_3 = e_3\,, \\ &e_4e_1 = e_4\,, \quad e_1e_4 = 0\,, \quad e_3e_2 = e_3\,, \quad e_2e_3 = 0\,. \end{aligned}$$

The fundamental representation of the groupoid algebra is supported in the 2-dimensional complex space $\mathcal{H} = \mathbb{C}^2$ with canonical basis $|+\rangle$, $|-\rangle$. The groupoid elements are represented by operators acting on the canonical basis as:

$$A_+|+\rangle = \pi(1_+)|+\rangle = |+\rangle\,, \qquad A_- = \pi(1_-)|-\rangle = 0\,,$$

that is with associated matrix:

$$A_+ = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Similarly we get:

$$A_- = \pi(1_-) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_\alpha = \pi(\alpha) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad A_{\alpha^{-1}} = \pi(\alpha^{-1}) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

Thus the groupoid algebra can be naturally identified with the algebra of 2×2 complex matrices $M_2(\mathbb{C})$ and the fundamental representation is just provided by the matrix-vector product of matrices and 2-component column vectors of $\mathbb{C}^2$.

Amplitudes are maps $f\colon \mathbf{G}_{\mathscr{A}} \to \mathbb{C}$, thus, they assign an amplitude to any of the transitions above, in particular we get $f(\alpha) = \langle - | A_f | + \rangle$, with A_f the operator associated to f.

Observables correspond to elements in the dual space of the algebra of the groupoid that we will identy again with the algebra of 2×2 complex matrices using the standard trace inner product, that is $\langle A, B \rangle = \mathrm{Tr}\,(A^\dagger B)$. Then real observables can be identified with 2×2 Hermitean matrices:

$$A = \begin{bmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{bmatrix} = x_0\,\mathbf{I} + \mathbf{x} \cdot \boldsymbol{\sigma} = \langle x, \sigma \rangle\,, \tag{19}$$

where σ_μ denote the standard Pauli σ-matrices:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

together with $\sigma_0 = \mathbf{I}$, and $\mathbf{x}$ is the vector in $\mathbb{R}^3$ with components (x_1, x_2, x_3). Then, the real observable f defined by the Hermitean matrix A above, Eq. (19), is given by:

$$f(1_+) = x_0 + x_3\,, \quad f(1_-) = x_0 - x_3\,, \quad f(\alpha) = x_1 + ix_2\,, \quad f(\alpha^{-1}) = x_1 - ix_2\,.$$

States ρ are defined as normalised positive functionals in $M_2(\mathbb{C})$, and they can be identified with density matrices $\rho = \rho^\dagger$, $\mathrm{Tr}\,\rho = 1$, $\rho \geq 0$. In this representation the complete system of observables will be the operator σ_3, identified for instance with the third component S_z of the spin operator $\mathbf{S}$ of an electron. The outcomes of this operator would be its eigenvalues ± 1 (that we have represented by the symbols $+$ and $-$ respectively). Notice that in the symbolic notation used above, this observable f_3 would be defined as $f_3(1_+) = 1$, $f_3(1_-) = -1$ and zero otherwise.

Stern-Gerlach transitions will be obtained by considering another complete system of experimental setups. It is not completely obvious, but after a minute reflection we will arrive to the conclusion that any other such complete system, call it $\mathscr{B}$, will provide exactly two outcomes, we may denote them as $\{\rightarrow, \leftarrow\}$. The algebra of transitions will be generated by $1_\rightarrow$, $1_\leftarrow$, β and β^{-1}, with β the 'flip' transition from the event $\rightarrow$ to the event $\leftarrow$. The algebra of transitions generated by $\mathscr{B}$ will be isomorphic to the algebra of transitions generated by $\mathscr{A}$, this means that there is an isomorphism Φ from the C^*-algebra of 2×2 matrices into itself. This isomorphism Φ will necessarily have the form $\Phi(A) = UAU^\dagger$ with U a unitary operator[17]. Notice that in such case the image of $1_\to$ in the description provided by $\mathscr{A}$ will be given by $\Phi(1_b) = UA_+U^\dagger$. It is then clear that the extended singleton introduced here is equivalent to a qubit system.

Finally we may consider the most general Hamiltonian dynamic for the extended singleton. For that we may consider a general hamiltonian H provided by a Hermitean matrix :

$$H = \begin{bmatrix} h_0 + h_3 & h_1 - ih_2 \\ h_1 + ih_2 & h_0 - h_3 \end{bmatrix}$$

and the evolution equation (13) becomes:

$$\dot{f}_+ = i(f_{\alpha^{-1}} h_z - \bar{h}_z f_\alpha)\,, \tag{20}$$

$$\dot{f}_- = i(\bar{h}_z f_\alpha - f_{\alpha^{-1}} h_z)\,, \tag{21}$$

$$\dot{f}_\alpha = i((f_- - f_+)h_z - 2h_3 f_\alpha)\,, \tag{22}$$

$$\dot{f}_{\alpha^{-1}} = i((f_+ - f_-)\bar{h}_z - 2h_3 f_\alpha)\,. \tag{23}$$

with $h_z = h_1 + ih_2$. Notice that $d(f_+ + f_-)/dt = 0$ and $f_+ + f_-$ is preserved. In particular if f where a density operator $\hat{\rho}$ the trace would be preserved (and equal to 1).

If $h_z = 0$, that means if H is diagonal, then $\dot{f}_\pm = 0$ and $f_\pm$ does not change. If we had a classical state, that is $p = p_+ 1_+ + p_- 1_-$, $p_+ + p_- = 1$, $p_\pm \geq 0$, then, for H diagonal there will be no evolution of the classical state.

Another interesting situation happens when $h_3 = 0$ and h_z is imaginary of the form $h_z = i\nu$, $\nu > 0$. Then, if $f_- f_+ > 0$, we have $f_\alpha(t) \to 0$ as $t \to \infty$, thus interpreting f as measuring the amplitude of the transition α, in the limit of t large, such amplitude vanish.

In the particular instance of a classical state defined by the density operator:

$$\hat{\rho} = \begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix},$$

we obtain for a hamiltonian of the form

$$h_\epsilon = i\epsilon\frac{\gamma}{2}(\delta_\alpha - \delta_{\alpha^{-1}})\,, \qquad \gamma > 0\,,$$

that corresponds to the case $h_0 = h_1 = h_2 = 0$ and $h_2 = i\epsilon\gamma/2$, and then:

$$\frac{d}{dt}\hat{\rho} = \epsilon \begin{bmatrix} 0 & (p_1 - p_2)\gamma/2 \\ (p_2 - p_1)\gamma/2 & 0 \end{bmatrix},$$

but, applying the classical transition described in Sect. 6.3, we obtain that the kernel k has only one entry k_{12} (where we are labelling now the events $+,-$ as

[17] A harder problem is when we are not considering complete descriptions, then, the map between both algebras will be just positive and we will use Choi's characterization of such transformations.

1, 2), and then the 2×2 Markovian matrix K in Eq. (17) becomes:

$$K = \begin{bmatrix} -\gamma & \gamma \\ \gamma & -\gamma \end{bmatrix} ,$$

and the classical dynamics of the state becomes:

$$\dot{p}_1 = -\gamma p_1 + \gamma p_2 \,, \qquad \dot{p}_2 = \gamma p_1 - \gamma p_2 \,.$$

7.2. **The harmonic oscillator.**

7.2.1. *The diagram K_∞.* We will now discuss a family of genuinely infinite-dimensional examples. Their kinematical description is as follows. The events are labelled by the symbols a_n $n = 0, 1, 2, ...$, and the groupoid structure is generated by a family of transitions $\alpha_n \colon a_n \to a_{n+1}$ for all n.

The assignment of physical meaning to the events a_n and the transitions α_n, that is, the identification of events with outcomes of a certain observable and the observation of physical transitions depends on the specific system under study. This in turn implies an assignment of physical meaning to the observables and the identification of the dynamics, and fixing the experimental setting chosen by the observers. For instance the events can be identified with the energy levels of a given system, an atom for instance, or the number of photons of a given frequency on a cavity. In the case of atoms the transitions will correspond to the physical transitions observed by measuring the photons emitted or absorbed by the system. In the case of an e.m. field on a cavity, the transitions will correspond to the change in the number of photons that could be determined by counting the photons emitted by the cavity or pumping a determined number of photons into it.

At this point, no specific values have been assigned to the events a_n yet, they just represent a sort of kinematical background for the theory. An assignment of numerical values to the events will be part of the dynamical prescription of the system. For instance, in the case of energy levels, we will be assigning a real number E_n to each event while in the case of photons, it will be a certain collection of non-negative integers $n_1, n_2, ...$ In what follows we will focus on the simplest non-trivial assignment of the number n to the event a_n.

A diagram describing this situation is shown in Fig. 2.

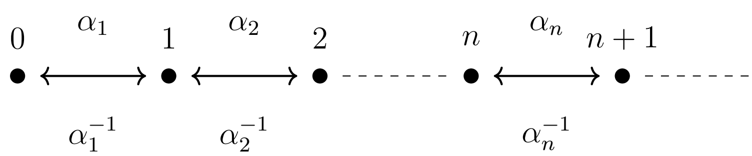

FIGURE 2. The diagram K_∞ generating the quantum harmonic oscillator.

The groupoid of transitions generated by this system $\mathbf{G}_{\mathscr{A}}$ is the groupoid of pairs of the natural numbers, that is, the complete graph with countable many vertices K_∞. Transitions $m \to n$ will be denoted by $\alpha_{n,m}$ or just (n,m) for short. The notation in the picture corresponds to $\alpha_n := \alpha_{n+1,n} = (n+1,n)$. With this notation, two transitions (n,m) and (j,k) are composable if and only if $m = j$, and their composition will be $(n,m) \circ (m,k) = (n,k)$. Notice that $(n,m)^{-1} = (m,n)$ and $1_n = (n,n)$ for all $n \in \mathbb{N}$.

The algebra of observables of the system will be given by functions on the groupoid $\mathbf{G}_{\mathscr{A}}$ but this time, in order to construct a C^*-algebra structure, we should start by considering first the set of functions which are zero except on a finite number of transitions and then take the closure with respect to an appropriate topology. Thus, denote by $\mathcal{F}_{\mathrm{fin}}(\mathbf{G}_{\mathscr{A}}) = \mathcal{F}_{\mathrm{fin}}(K_\infty)$ the set of functions on K_∞ which are zero except on a finite number of pairs (n,m). We may write any one of these functions as:

$$f = \sum_{n,m=1}^{\infty} f(n,m)\delta_{(n,m)}\,, \tag{24}$$

where only a finite number of coefficients $f(n,m)$ are different from zero (the function $\delta_{(n,m)}$ is the obvious function $\delta_{(n,m)}(\alpha_{jk}) = \delta_{nj}\delta_{mk}$). We can define as usual the convolution product on $\mathcal{F}_{\mathrm{fin}}(K_\infty)$:

$$(f \star g)(n,m) = \sum_{(n,j)\circ(j,m)=(n,m)} f(n,j)g(j,m) = \sum_j f(n,j)g(j,m)\,.$$

Hence, using Heisenberg's interpretation of observables as (infinite) matrices, we may consider the coefficients $f(n,m)$, $n,m = 0,1,\ldots$, in the expansion (24) of the observable f as defining an infinite matrix F whose entries F_{nm} are $f(n,m)$, and the convolution product on the algebra $\mathcal{F}_{\mathrm{fin}}(K_\infty)$ is just the matrix product of the matrices F and G corresponding to f and g respectively (notice that the product is well defined as there are only finitely many non zero entries on both matrices). The involution $f \mapsto f^*$ is defined in the standard way $f^*(n,m) = \overline{f(m,n)}$ for all n,m.

The fundamental representation of the system will be supported on the Hilbert space $\mathcal{H}$ generated by vectors $|n\rangle$, $n = 0,1,\ldots$, that is, the family of vectors $\{|n\rangle\}$ define an orthonormal basis of $\mathcal{H}$. Thus, the Hilbert space $\mathcal{H}$ is the space $l^2(\mathbb{Z})$ of infinite sequences $z = (z_0, z_1, z_2, \ldots)$ of complex numbers with $||z||^2 = \sum_{n=0}^\infty |z_n|^2 < \infty$. The fundamental representation π of the algebra $\mathcal{F}_{\mathrm{fin}}(K_\infty)$ is just given by[18]:

$$\pi(\alpha_{nm})|k\rangle = \delta_{mk}|n\rangle\,,$$

[18] With some abuse of notation as we are identifying the functions $\delta_{(n,m)}$ with the transition α_{nm}.

that is, $\pi(\alpha_{nm})$ is the operator in $\mathcal{H}$ that sends the vector $|m\rangle$ into the vector $|n\rangle$ and zero otherwise. Even in more concise terms: the fundamental representation of the transition α_n maps the vector $|n\rangle$ into the vector $|n+1\rangle$. In particular $\pi(\alpha_1)|0\rangle = |1\rangle$. Notice that $\pi(\alpha_n^{-1})|n+1\rangle = |n\rangle$.

Using the fundamental representation π we may define a norm on $\mathcal{F}_{\mathrm{fin}}(K_\infty)$ as $||f|| = ||\pi(f)||_{\mathcal{H}}$ and consider the completion $\mathcal{F}(K_\infty)$ of $\mathcal{F}_{\mathrm{fin}}(K_\infty)$ with respect to it. It is now clear that such completion is a C^*-algebra as $||f^*\star f|| = ||\pi(f^*\star f)||_{\mathcal{H}} = ||\pi(f^*)\pi(f)||_{\mathcal{H}} = ||\pi(f)^\dagger\pi(f)||_{\mathcal{H}} = ||\pi(f)||^2_{\mathcal{H}} = ||f||^2$. Moreover, by construction, the representation π is continuous and has a continuous extension to the completed algebra $\mathcal{F}(K_\infty)$. By construction the map π defines an isomorphism of algebras between the algebra $\mathcal{F}(K_\infty)$ and the algebra $\mathcal{B}(\mathcal{H})$ of bounded operators on the Hilbert space $\mathcal{H}$[19]. The elementary transitions α_n generating the graph K_∞ contain the relevant information of the system. Any transition α_{nm} can be obtained composing elementary transitions: $\alpha_{nm} = \alpha_n\alpha_{n+1}\cdots\alpha_{m-1}$ $(n<m)$.

7.2.2. *The standard harmonic oscillator.* From the considerations raised in Sect. 7.2.1, once we have chosen the assignment $a_n = n$, $n = 0, 1, 2, \ldots$, we may define the functions a and $a^\dagger$ in $\mathcal{F}(K_\infty)$ as follows:

$$a(\alpha_n) = \sqrt{n}\,, \qquad a^\dagger(\alpha_n) = \sqrt{n+1}\,, \tag{25}$$

or, in terms of the algebra of the groupoid K_∞, we will have that a and $a^\dagger$ are given as the formal series:

$$a = \sum_{n=0}^{\infty} \sqrt{n}\,\alpha_n^{-1}\,, \qquad a = \sum_{n=0}^{\infty} \sqrt{n+1}\,\alpha_n\,.$$

Strictly speaking a, $a^\dagger$ are not elements in the groupoid algebra $\mathcal{F}(K_\infty)$, indeed, they define unbounded operators in the fundamental representation, however, they do define functions on K_∞ and we can manipulate them formally. A simple computation shows that:

$$[a, a^\dagger] = \mathbf{1}\,,$$

with $\mathbf{1} = \sum_{n=0}^{\infty} 1_n$, or in terms of functions in K_∞, $[a,a^\dagger](1_n) = 1$ for all n and zero otherwise. Hence we may construct the Hamiltonian function:

$$h = a^\dagger a + \frac{1}{2} = \sum_{n=0}^{\infty} n\delta_n + \frac{1}{2}\,.$$

and the corresponding equations of motion:

$$\dot{a} = i[a,h] = -ia\,, \qquad \dot{a}^\dagger = i[a^\dagger, h] = ia^\dagger\,,$$

which constitute the standard equations of motion for the quantum harmonic oscillator.

[19]This constitutes a particular instance of Renault's construction of C^*-algebras defined by groupoids [19], see also the comments in [21].

Notice that the functions a, $a^\dagger$ on the groupoid K_∞ define densely defined unbounded operators on the Hilbert space $\mathcal{H} = l^2(\mathbb{Z})$ supporting the fundamental representation such that $\pi(a)^\dagger = \pi(a^\dagger)$. Moreover the Hamiltonian operator $H = \pi(h)$ may be identified with the Hamiltonian operator of a harmonic oscillator with creation and annihilation operators $\pi(a^\dagger)$ and $\pi(a)$ respectively.

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