SCHWINGER'S PICTURE OF QUANTUM MECHANICS III: THE STATISTICAL INTERPRETATION[1]

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Abstract. Schwinger's algebra of selective measurements has a natural interpretation in the formalism of groupoids. Its kinematical foundations as well as the structure of the algebra of observables of the theory was presented in [Ib18a, Ib18b]. In this paper a closer look to the statistical interpretation of the theory is taken and it is found that a natural interpretation in terms of Sorkin's quantum measure emerges naturally. It is proven that a suitable class of states of the algebra of virtual transitions of the theory define quantum measures by means of their corresponding decoherence functionals. Quantum measures satisfying a reproducing property are described and a class of states, called local states, possessing the Dirac-Feynman 'exponential of the action' form are characterized. Finally, Schwinger's transformation functions are interpreted similarly as transition amplitudes defined by suitable states. The simple example of the qubit and the double slit experiment are described in detailed illustrating the main aspects of the theory.

Contents

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[1]THIS IS A DRAFT. NOT THE FINAL VERSION YET.

1. Introduction

1.1. On the probabilistic interpretation of quantum mechanics by Feynam and Schwiger. A careful interpretation of the probabilistic nature of Quantum Mechanics led both J. Schwinger and R.P. Feynman to their own formulations of the theory. R. P. Feynman forcefully stated that "...it seems worthwhile to emphasize the fact that [the observed experimental facts] are all simply direct consequences of Eq. (5),

$$\varphi_{ab} = \sum_b \varphi_{ab}\varphi_{bc} \,, \tag{1}$$

for it is essentially this equation that is fundamental in my formulation of quantum mechanics" [Fe48]. The quantum amplitudes φ_{ab} being such that $|\varphi_{ab}|^2$ represent the classical probability that if measurement A gave the result a, then measurement B will give the result b. Then Feynman, following Dirac's powerful insight [Di33], proceeded by postulating that this quantum probability amplitude "has a phase proportional to the action" and implemented his sum over histories description of quantum mechanics [Fe48] that we, together with Yourgrau and Mandelstan, "...cannot fail to observe that Feynman's principle – and this is no hyperbole – expresses the laws of quantum mechanics in an exemplary neat and elegant manner" [Yo68, Footnote 6].

Alternatively, J. Schwinger introduced the statistical interpretation of his selective measurement symbols stating: "...measurements of properties B, performed on a system in a state c' that refers to properties incompatible with B, will yield a *statistical distribution* of the possible values. Hence, only a determinate fraction of the systems emerging from the first stage will be accepted by the second stage. We express this by the general multiplication law:

$$M(a', b')M(c', d') = \langle b' \mid c'\rangle M(a', d') \,,$$

where $\langle b' \mid c' \rangle$ is a number characterising the statistical relation bewteen states b' and c'" [Sc91, Chap. 1.3]. We will just add here that Schwinger's transformation functions $\langle b' \mid c' \rangle$ played an instrumental role in the development of the theory [Sc51].

Much more recently R. Sorkin, in his paper presenting a quantum measure interpretation of quantum mechanics [So94], commenting on the standard statistical interpretation of quantum mechanics stressed: "...to the untutored mind, however, the formal rules of the path-integral scheme, could seem unnatural and contrived. Why are probabilites squares of amplitudes...?".

In this paper we will try to offer an new interpretation of these ideas, tying together the (apparently) disparate statistical interpretations of quantum mechanics upon which Schwinger and Feynman founded their own descriptions of the theory, and putting them under the unifying conceptual framework provided by Sorkin's quantum measure interpretation of quantum mechanics, by using the recently proposed groupoid interpretation of Schwinger's algebra of measurements in [Ib18a, Ib18b]. It will be shown that the results are equivalent the usual formulations (or better stated, in particular instances, they reproduce both Schwinger's and Feynman's interpretations), then there are, therefore, no fundamentally new results but, as Feynman's himself stated in the introduction to its epoch making paper [Fe48]: "... However, there is a pleasure in recognizing old things from a new point of view. Also, there are problems for which the new point of view offers a distinct advantage".

In our previous works [Ib18a, Ib18b] both the kinematical background and the basic dynamical structures for a new description of quantum mechanical systems inspired on Schwinger's algebra of measurement were presented. It was argued that the basic kinematical structure to describe a theory of quantum systems can be developed from the notions of *events*, *transitions* and *transformations* that, from a mathematical viewpoint, satisfy the algebraic properties of a 2-groupoid. 'Events'[1] and 'transitions' provide a natural abstract setting for Schwinger's notion of physical selective measurements and form a groupoid from the mathematical perspective.

The concept of 'event' extends the notion of 'state' used by Schwinger (that in his case coincides with the standard notion of maximal compatible family of measurements): "... a complete measurement, which is such that the systems chosen possess definite values for the maximum number of attributes... Thus the optimum state of knowledge concerning a given system is realized by subjecting it to a *complete selective measurement*" [Sc91, Chap. 1.2]. We would like to extend such notion to consider possible outcomes of observations or manipulations

[1]The name 'event' was chosen for the lack of a better word. Schwinger's called them 'states', however 'state' will be used in a widely extended technical sense that, at the same time, captures perfectly well the required statistical meaning. Thus we will stick with 'events' for the time being.

performed on a given system, not necessarily complete in any sense so that we may consider histories of events as natural ingredients of the theory (a possibility already anticipated by Feynman: ".... Suppose a measurement is made which is capable only of determining that the path lies somewhere within a region R. The measurement is to be what we might call an 'ideal measurement'... I have not been able to find a precise definition", [Fe48]).

On the other hand, in Schwinger's conceptualisation, the notion of 'transition' is clearly identified and corresponds to 'measurements that change the state' [Sc91, Chap. 1.3]. Thus the notion of transition introduced in [Ib18a] extends the notion of selective measurements that change the state to include all physical feasible changes between events of the system. Transitions though can be naturally composed and their composition law satisfies the axioms of a groupoid which constitutes the primary algebraic structure of the theory.

Instrumental on all of it is the assumption that transitions are invertible. In this sense we agree with Feynman when states quite forcefully: "The fundamental (microscopic) phenomena in nature are symmetrical with respect to interchange of past and future" [Fe05, Chap. I, p. 3]. We share this principle, that leads to the assumption that the basic of the theory transitions must be invertible[2].

Passing from a 'reference system' with outcomes or events denoted by a and transitions α to another with events b and transitions β requires a theory of transformations. Such theory is developed by Schwinger [Sc91, Chap. 2.5] reproducing the standard theory of unitary operators developed by Dirac. However, previous to that and at a more basic level, Schwinger introduces the notion of *transformation function* $\langle a' \mid b' \rangle$, a notion that will be discussed in the present paper: "...measurement symbols of one type can be expressed as linear combinations of the measurement symbols of another type. From its role in effecting such connections the totality of numbers $\langle a' \mid b' \rangle$ is called the *transformation function* relating the $a-$ and the $b-$descriptions". In Schwinger's formulation, the transformation functions arise as a concrete representation (of a abstract operation that was coined 'transformation' and that affects to specific transitions as explained in [Ib18a]). The theory of 'transformations' thus developed fits naturally in the algebraic setting of the theory of groupoids and determines a 2-groupoid structure on top of Schwinger's groupoid of events and transitions.

The previous ideas fix the kinematic framework of the theory as discussed in [Ib18a] where no attempt to introduce a dynamical content was made. In this sense we were following R. Sorkin's dictum of "proposing a framework in which

[2]However, Schwinger, even if his formalism implies that the selective measurements that change the state are 'invertible', only reluctantly acknowledges that when states: "...the arbitrary convention that accompanies the interpretation of the measurement symbols and their products - the order of events is read from right to left (sinistrally), but any measurement symbol equation is equally valid if interpreted in the opposite sense (dextrally), and no physical result should depend upon which convention is employed" [Sc91, Chap. 1.7].

the ontology or 'kinematics' and the dynamics or 'laws of motion' are as sharply separated from each other as they are in classical physics" [So94].

The dynamical aspects of the theory where discussed though in [Ib18b]. The departing point of the analysis was the key idea of considering an observable as the assignment of an amplitude to any transition, that is, an observable is a function on the groupoid of transitions, idea which just reflects the abstraction of the determination of an observable by means of the amplitudes $\langle a|A|a'\rangle$. The notion of observables thus introduced leads in a natural way to the construction of the C^*-algebra of observables of the system and a Heisenberg-like formulation of dynamics as infinitesimal generators of automorphisms of it.

Physical states of the system correspond in this way to states of the C^*-algebra of observables, that is, of normalised linear positive functionals on the algebra, thus opening the way to the interpretation of the theory in terms of Hilbert spaces and operators by applying the GNS construction associated to any state of the theory. This idea will be used repeatedly in the present paper to provide a sound statistical interpretation to it.

We must stress here that this approach departs from Schwinger's derivation of the laws of dynamics from a quantum dynamical principle, that nevertheless will be undertaken in a future publication [Ib18d]. We consider that the approach taken in [Ib18b] is more natural and we agree with R. Sorkin: "...quantum theory differs from classical mechanics in its dynamics, which ... is stochastic rather than deterministic. As such the theory functions by furnishing probabilities for sets of histories" [So94]. In this sense the dynamical theory that we propose is closer in spirit to R. Sorkin's understanding of quantum mechanics as a quantum measure theory, point of view that will be one of the main subjects of the present paper.

Thus the present paper will provide a statistical interpretation of the theory by constructing a quantum measure in Sorkin's sense [So94] on the groupoid of transitions associated to any invariant state of the algebra of transitions of the theory. The key idea to do that is by realising that an state ρ, determines a function φ of positive semidefinite type on the given groupoid, and this function actually defines a decoherence functional. The relation between decoherence functionals and quantum measures allows to provide the desire statistical interpretation of the theory, identifying the amplitudes of transitions with the values $\varphi(\alpha)$ of the positive semidefinite function defining the decoherence functional and its module square representing the 'probability' of an event. In developing the theory, the GNS construction will be used to interpret the obtained notions in the more familiar terms of vector-valued measures on Hilbert spaces, and a extension of Naimark's reconstruction theorem for groupoids will be proved.

The second part of the part will be devoted to identify two classes of states that have a specific physical meaning. The first one are those states such that the associated physical amplitudes satisfy Feynman's reproducing property stated above (1). We will characterise those states in a purely algebraic way as those whose characteristic functions φ are idempotent with respect to the convolution product in the algebra of observables of the theory.

Another of the purposes of the paper, in order to offer a proper and complete account of all ideas involved in Sorkin's quantum measure notion, would be to understand the singularity of Dirac-Feynman postulate that amplitudes have the form $e^{\frac{i}{\hbar}S}$ for a quantum theory on space-time. It will be shown that there is again a purely algebraic notion in the groupoids setting for quantum mechanics that characterises completely such states. Such notion have been called 'locality' as it abstracts the notion of locality in a space-time based theory. The proof of the corresponding theorem is worked out in detail in the finite-dimensional case and it constitutes one of the main results reported here.

Finally, the third part of the paper is devoted to put Schwinger's theory of transformations functions in the same footing as the previous notions. This is achieved by observing that there are natural states, those associated to outcomes a of the theory, whose corresponding amplitudes on transitions associated to events b on different frames provide precisely Schwinger's transitions functions $\langle b|a\rangle$ and they are given precisely as inner products of vectors on suitable Hilbert spaces (again obtained by a natural use of the GNS construction).

The rest of this introduction will be devoted to succinctly summarise the basic notions and notations on groupoids and their algebras used throughout the paper (see the preceding papers [Ib18a], [Ib18b] for a detailed account of these ideas).

1.2. **The groupoids description of Schwinger's algebra of measurements: basic notations and definitions.** Even if groupoids can be described in a very abstract setting using category theory, in this paper we will use set-theoretical concepts and notations instead to work with them. For the most part we will assume that groupoids are discrete (countable) or even finite[3].

Thus a groupoid $\mathbf{G}$ will be a set, whose elements denoted by greek letters $\alpha, \beta, \gamma, ...$ will be called transitions. There are two maps $s, t \colon \mathbf{G} \to \Omega$, called respectively source and target, from the groupoid $\mathbf{G}$ into a set Ω whose elements will be denoted by lowercase latin letters $a, b, c, \ldots, x, y, z$ and called events. We will often use the diagrammatic representation $\alpha \colon a \to a'$ for the transition α if $s(\alpha) = a$ and $t(\alpha) = a'$. Notice that the previous notation doesn't imply that α is a map from a set a into another set a', even if occasionally we will use the notation $\alpha(a)$ to denote $a' = t(\alpha)$. We will also say that the transitions α *relates* the event a to the event a'.

Denoting by $\mathbf{G}(y, x)$ the set of transitions relating the event x with the event y, i.e., $\alpha \in \mathbf{G}(y, x)$ if $\alpha \colon x \to y$, there is a composition law $\circ \colon \mathbf{G}(z, y) \times \mathbf{G}(y, x) \to$

[3]We will be concerned mostly with the algebraic structures of the theory leaving many of the deep and delicate analytical details involved in the infinite dimensional setting for further discussion.

$\mathbf{G}(z, x)$, such that if $\alpha\colon x \to y$ and $\beta\colon y \to z$, then $\beta \circ \alpha\colon x \to z$[4]. Two transitions α, β such that $t(\alpha) = s(\beta)$ will said to be composable. The set of composable transitions form a subset of the Cartesian product $\mathbf{G} \times \mathbf{G}$ sometimes denoted by $\mathbf{G}_2$.

It is postulated that the composition law $\circ$ is associative whenever the composition of three transitions makes sense, that is: $\gamma \circ (\beta \circ \alpha) = (\gamma \circ \beta) \circ \alpha$, whenever $\alpha\colon v \to x$, $\beta\colon x \to y$ and $\gamma\colon y \to z$. Any event $a \in \Omega$ has associated a transition denoted by 1_a satisfying the properties $\alpha \circ 1_a = \alpha$, $1_{a'} \circ \alpha = \alpha$ for any $\alpha\colon a \to a'$. Notice that the assignment $a \mapsto 1_a$ defines a natural inclusion $i\colon \Omega \to \mathbf{G}$ of the space of events in the groupoid $\mathbf{G}$. Finally it will be assumed that any transition $\alpha\colon a \to a'$ has an inverse, that is there exists $\alpha^{-1}\colon a' \to a$ such that $\alpha \circ \alpha^{-1} = 1_{a'}$, and $\alpha^{-1} \circ \alpha = 1_a$.

Given an event $x \in \Omega$, we will denote by $\mathbf{G}_+(x)$ the set of transitions starting at a, that is, $\mathbf{G}_+(x) = \{\alpha\colon x \to y\} = s^{-1}(x)$. In the same way we define $\mathbf{G}_-(y)$ as the set of transitions ending at y, that is, $\mathbf{G}_-(y) = \{\alpha\colon x \to y\} = t^{-1}(y)$. The intersection of $\mathbf{G}_+(x)$ and $\mathbf{G}_-(x)$ consists of the set of transitions starting and ending at x and is called the isotropy group G_x at x: $G_x = \mathbf{G}_+(x) \cap \mathbf{G}_-(x)$.

Given an event $a \in \Omega$, the orbit $\mathcal{O}_a$ of a is the subset of all events related to a, that is, $a' \in \mathcal{O}_a$ if there exists $\alpha\colon a \to a'$. The isotropy groups G_x and G_y of two events in the same orbit, $x, y \in \mathcal{O}_a$, are isomorphic. Clearly the isotropy group G_a acts on the right on the space of transitions leaving from a, that is, there is a natural map $\mu_a\colon \mathbf{G}_+(a) \times G_a \to \mathbf{G}_+(a)$, given by $\mu_a(\alpha, \gamma_a) = \alpha \circ \gamma_a$ (notice that the transition $\gamma_a\colon a \to a$ doesn't change the source of $\alpha\colon a \to a'$). Then it is easy to check that there is a natural bijection between the space of orbits of G_a in $\mathbf{G}_+(a)$ and the elements in the orbit $\mathcal{O}_a$, given by $\alpha \circ G_a \mapsto \alpha(a) = a'$. Then we may write:

$$\mathbf{G}_+(a)/G_a \cong \mathcal{O}_a \, .$$

It is obvious that there is also a natural left action of G_a into $\mathbf{G}_-(a)$ and that $G_a \backslash \mathbf{G}_-(a) \cong \mathcal{O}_a$ too. We will say that the groupoid is connected or transitive if it has is a single orbit, $\Omega = \mathcal{O}_a$, for some a. Then it can be proved that $\Omega = \mathcal{O}_x$ for any $x \in \Omega$. Any groupoid decomposes as the disjoint union of connected groupoids, any of them the restriction of the given groupoid to any one of its orbits. In what follows we will always assume that groupoids are connected.

If the groupoid $\mathbf{G}$ is finite the groupoid algebra, or algebra of virtual transitions, of the groupoid $\mathbf{G}$ is defined in the standard way as the associative algebra $\mathbb{C}[\mathbf{G}]$ generated by the elements of $\mathbf{G}$ with relations provided by the composition law of the groupoid, that is, elements $\mathbf{a}$ in $\mathbb{C}[\mathbf{G}]$ are finite formal linear combinations $\mathbf{a} = \sum_{\alpha \in \mathbf{G}} a_\alpha \, \alpha$, with a_α complex numbers. The groupoid algebra elements $\mathbf{a}$ can be though as generalized or mixed transitions for the system and will be called

[4]The 'backwards' notation for the composition law has been chosen so that the various representations and compositions used along the paper look more natural, it is also in agreement with the standard notation for the composition of functions.

also virtual transitions. The associative composition law on $\mathbb{C}[\mathbf{G}]$ is defined as:

$$\mathbf{a} \cdot \mathbf{a}' = \sum_{\alpha,\alpha' \in \mathbf{G}} a_\alpha a_{\alpha'} \, \delta_{\alpha,\alpha'} \, \alpha \circ \alpha' = \sum_{\alpha,\alpha' \in \mathbf{G}_2} a_\alpha a_{\alpha'} \, \alpha \circ \alpha' \, ,$$

where the indicator function $\delta_{\alpha,\alpha'}$ takes the value 1 if α and α' are composable, and zero otherwise. The groupoid algebra has a natural antilinear involution operator denoted $*$, defined as $\mathbf{a}^* = \sum_\alpha \bar{a}_\alpha \, \alpha^{-1}$, for any $\mathbf{a} = \sum_\alpha a_\alpha \, \alpha$.

If the groupoid $\mathbf{G}$ is finite, there is a natural unit element $\mathbf{1} = \sum_{a \in \Omega} 1_a$ in the algebra $\mathbb{C}[\mathbf{G}]$[5].

Another family of relevant mixed transitions are given by $\mathbf{1}_{G_a} = \sum_{\gamma_a \in G_a} \gamma_a$, which are the characteristic 'functions' of the isotropy groups G_a and $\mathbf{1}_{\mathbf{G}_\pm(a)} = \sum_{\alpha \in \mathbf{G}_\pm(a)} \alpha$ that represent the characteristic 'functions' of the sprays $\mathbf{G}_\pm(a)$ at a. Finally, we should mention the 'incidence' or total transition, also called the 'incidence matrix' of the groupoid, defined as $\mathbb{I} = \sum_\alpha \alpha$.

[5]In any case we always consider that the algebra of transitions of the system to be unital by the standard procedure of adding a unit to $\mathbb{C}[\mathbf{G}]$.

2. Quantum measures and Schwinger's algebra

2.1. **Quantum measures and decoherence functionals.** Sorkin's introduction of the notion of a quantum measure allows for an statistical interpretation of Quantum Mechanics without recurring to some of the standard difficulties related to the existence of observers to assess the predictive capacity of the theory or the collapse of the state of the system [So94].

According to Sorkin's theory Quantum Mechanics can be understood as a generalized measure theory on the space $\mathcal{S}$ of all possible histories of some physical system. It assigns a non-negative real number $\mu(A)$, the quantum measure of A, to every measurable subset A of the set of histories of the system. The quantum measure μ is not an ordinary probability measure because in general the interference term:

$$I_2(A,B) = \mu(A \sqcup B) - \mu(A) - \mu(B)\,, \tag{2}$$

for two disjoint sets A, B, doesn't vanish. Thus the feature that distinguishes a quantum theory from a classical one is interference. This means that the quantum measure μ will enjoy different formal properties than classically. We can define the following family of set-functions for any generalized measure theory over a sample space $\mathcal{S}$ equipped with a σ-algebra Σ of measurable sets:

$$\begin{aligned} I_1(A) &= \mu(A)\,, \\ I_2(A,B) &= \mu(A\sqcup B) - \mu(A) - \mu(B)\,, \\ I_3(A,B,C) &= \mu(A\sqcup B\sqcup C) - \mu(A\cup B) - \mu(A\sqcup C) - \mu(B\sqcup C) \\ &\quad +\mu(A) + \mu(B) + \mu(C)\,, \end{aligned} \tag{3}$$

and so on, where A,B,C are disjoint sets of $\mathcal{S}$. Higher order interference relations beyond bipartite and tripartite interference terms as given by Eqs. (2), (3), can be defined as:

$$\begin{aligned} I_n(A_1,\ldots,A_n) &= \mu(A_1\sqcup\cdots\sqcup A_n) - \sum_{i_1<i_2<\cdots<i_{n-1}} \mu(A_{i_1}\sqcup\cdots\sqcup A_{i_{n-1}}) \\ &\quad + \sum_{i_1<i_2<\cdots<i_{n-2}} \mu(A_{i_1}\sqcup\cdots\sqcup A_{i_{n-2}}) + \cdots + (-1)^n \sum_{i=1}^{n} \mu(A_i)\,, \end{aligned}$$

for any family of disjoint sets A_i. It can be shown that the interference relation I_n of order n implies I_r for all $r \geq n$. Actually it is easy to prove by induction that:

$$\begin{aligned} I_{n+1}(A_0,A_1,\ldots,A_n) &= I_n(A_0\sqcup A_1, A_2,\ldots,A_n) \\ &\quad - I_n(A_0,A_2,\ldots,A_n) - I_n(A_1,A_2,\ldots,A_n)\,, \end{aligned} \tag{4}$$

hence, if $I_n = 0$ on any family of disjoint measurable sets A_i, then I_{n+1} will vanish too.

The interference functions I_n allow us to distinguish between different types of theories according to their statistical properties. According to Sorkin a theory is of grade-k if it satisfies $I_{k+1} = 0$. Thus a classical measure theory is a grade-1 measure theory, which is equivalent to saying that there is no bipartite interference: $\mu(A \sqcup B) = \mu(A) + \mu(B)$, and Kolmogorov's standard probability interpretation of the measure $\mu(A)$ can be used.

A quantum measure theory is a grade-2 measure theory, that is, a quantum measure is a set function $\mu \colon \Sigma \to \mathbb{R}^+$ such that it satisfies the grade-2 additivity condition $I_3 = 0$[6]. Thus a quantum measure allows to describe a theory with non-trivial second order (but no higher order) vanishing interference $I_k = 0$, $k \geq 3$, however the statistical interpretation of such condition must be assessed because the classical probabilistic interpretation of the measure of a set as a frequency of outcomes of a random variable cannot be hold anymore as it is easily exemplified by the double slit experiment[7].

The quantum measure of an event in Sorkin's sense is not simply the sum of the probabilities of the histories that compose it, but is given (in an extension of Born's rule) by the sum of the squares of certain sums of the complex amplitudes of the histories which comprise the event[8].

More important to our interest in this paper is to understand the construction of quantum measures in the abstract background provided by the groupoid interpretation of the fundamental algebraic properties of quantum systems. For this purpose we will discuss the relation of quantum measures and decoherence functionals in the realm of groupoids extending in this way recent results on their representations that will be helpful in providing a new statistical interpretation of Schwinger's transformation functions. It should be pointed out that the recursive relation Eq. (4) applied to the additivity condition $I_3 = 0$, implies that I_2 is additive for disjoint sets A, B, C, that is:

$$I_2(A \sqcup B, C) = I_2(A, C) + I_2(B, C) \,,$$

In fact, we get that if I_2 where additive on the first factor for all C (not just for C's disjoint with A and B), then spanning $I_2(A \sqcup B, A \sqcup B)$ we will get $\mu(A) = \frac{1}{2} I_2(A, A)$ and the quantum measure could be recovered as a quadratic function on the algebra of measurable sets. This leads to consider biadditive set functions $D \colon \Sigma \times \Sigma \to \mathbb{C}$

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[6]Technically speaking this definition will correspond to a pre-quantum, or finite, quantum measure, being necessary to add a continuity condition to make it consistent with Σ being a σ-algebra, that is, $\lim \mu(A_i) = \mu(\bigcap A_i)$ for all decreasing sequences and $\lim \mu(A_i) = \mu(\bigcap A_i)$ for all increasing sequences, see [Gu09] for more details.

[7]It has been shown recently though that the tripartite interference condition $I_3 = 0$ holds in quantum mechanics by using a 3-slit experiment [Si09, Ne17].

[8]Without entering such discussion here, Sorkin has proposed an interpretation of the number assigned to an event A by a quantum measure in terms of the notion of 'preclusion' instead of the of notion of 'expectation' [So95]. Preclusion is related to the impossibility of null sets and in this context it is necessary to add a regularity condition to a quantum measure, that is $\mu(A) = 0$ implies that $\mu(A \cup B) = \mu(B)$ and $\mu(A) = 0$ implies that $\mu(B) = 0$ for all $B \in \Sigma$, $B \subset A$ - then the quantum measure is called completely regular.

to construct quantum measures. Actually, this idea is deeply rooted in the histories approach to quantum mechanics under the name of decoherence functionals [Ge90] and what is important for the arguments to follow, a significant class of normalised quantum measures can be built by using a decoherence functional D.

Thus a general decoherence functional on a measurable space (X, Σ) is a set function $D\colon \Sigma \times \Sigma \to \mathbb{C}$ such that:

$$D(A, B) = \overline{D(B, A)}\,, \qquad \forall A, B \in \Sigma\,, \tag{5}$$

$$D(A, A) \geq 0\,, \tag{6}$$

and

$$D(A \sqcup B, C) = D(A, C) + D(B, C)\,, \quad \forall A, B, C \in \Sigma\,, \quad A \cap B = \emptyset\,. \tag{7}$$

It will be assumed that the decoherence functional D is normalised, that is $D(X, X) = 1$[9], and if this notion is sufficient to construct a quantum measure by means of:

$$\mu(A) = D(A, A)\,. \tag{8}$$

However in order to obtain a continuous completely regular quantum measure (just a quantum measure for short in what follows) it is necessary to introduce a slightly more restrictive definition of decoherence functional [Do10], that is, a strongly positive normalised decoherence functional D is a complex-valued set function defined on the Cartesian product $\Sigma \times \Sigma$ such that:

i.- Normalization: $D(X, X) = 1$.
ii.- σ-Additivity: $D(\cdot, A)$ is a complex measure for any $A \in \Sigma$.
iii.- Positivity: Given any natural number n and any family $A_1, \ldots, A_n$ of measurable sets in Σ, then $D(A_i, A_j)$ is a positive semi-definite $n \times n$-matrix.

Condition (i) is an irrelevant normalisation condition. Notice that condition (iii) implies conditions (5) and (6) above, while condition (ii) implies the finite-additivity condition (7). Then it is a routine check to show that the set function μ defined by Eq. (8) is a completely regular quantum measure (see for instance [Gu09]).

2.2. **Quantum measures on groupoids.** As it turns out the groupoid formalism to describe quantum systems provides a natural framework to construct quantum measures and thus, it provides a statistical interpretation of the theory. Thus, we will consider that the connected discrete[10] groupoid $\mathbf{G} \rightrightarrows \Omega$ provides a complete description of our system.

[9] The notion of decoherence functional is known under the name of bimeasures in abstract measure theory and has been discussed thoroughly in multiple contexts, see for instance [Ib14] and references therein.

[10] As customary in this series we will assume that the groupoid is discrete countable or finite to avoid the technical complications brought by functional analysis, even though most of the theory can be extended naturally to continuous or Lie groupoids with ease.

2.2.1. *Decoherence functionals and positive semidefinite functions on groupoids.* Because of its σ-additivity a decoherence functional D on the discrete groupoid $\mathbf{G}$ is determined by their values on singletons[11], that is,

$$D(A,B) = \sum_{\alpha\in A,\beta\in B} D(\{\alpha\},\{\beta\}) \,,$$

then we may consider a decoherence functional on discrete groupoids as defined by a bivariate function $\Phi\colon \mathbf{G}\times\mathbf{G}\to\mathbb{C}$ such that:

(1) $\sum_{\alpha,\beta\in\mathbf{G}} \Phi(\alpha,\beta) = 1$.
(2) Given any natural number n and any family $\alpha_1,\dots,\alpha_n$ of transitions in $\mathbf{G}$, then $\Phi(\alpha_i,\alpha_j)$ is a positive semi-definite $n\times n$-matrix.

We will also say that the bivariate function Φ is positive semidefinite. Let us introduce a further relevant notion for the purposes of this paper. A function $\varphi\colon \mathbf{G}\to\mathbb{C}$ will be said to be positive semidefinite if for any $n\in\mathbb{N}$, $\xi_i\in\mathbb{C}$, $\alpha_i\in\mathbf{G}$, $i=1,\dots,n$, it is satisfied:

$$\sum_{i,j=1}^{n} \bar{\xi}_i\xi_j\,\varphi(\alpha_i^{-1}\circ\alpha_j)\geq 0\,,$$

where the sum is taken over all pairs α_i,α_j such that the composition $\alpha_i^{-1}\circ\alpha_j$ makes sense, that is $t(\alpha_j)=t(\alpha_i)$. If we want to emphasise that the sum is restricted to those pairs α_i and α_j such that α_i^{-1} and α_j are composable we will write:

$$\sum_{i,j=1}^{n} \bar{\xi}_i\xi_j\,\varphi(\alpha_i^{-1}\circ\alpha_j)\,\delta(t(\alpha_i),t(\alpha_j))\geq 0\,,$$

where the delta function $\delta(t(\alpha_i),t(\alpha_j))$ implements the composability condition above.

Clearly any positive semidefinite function φ on $\mathbf{G}$ defines a bivariate positive semidefinite function Φ by means of:

$$\Phi(\alpha,\beta)=\delta(t(\alpha),t(\beta))\,\varphi(\alpha^{-1}\circ\beta)\,,\qquad \alpha,\beta\in\mathbf{G}\,. \tag{9}$$

Among the decoherence functionals D on groupoids the invariant ones play a distinguished role. A decoherence functional D on the groupoid $\mathbf{G}$ is said to be (left-) invariant if $D(\alpha\circ A,\alpha\circ B)=D(A,B)$ for all subsets A,B.

In terms of the corresponding bivariate positive semidefinite function Φ, a decoherence functional is invariant iff Φ is invariant with respect to the natural action of the groupoid $\mathbf{G}$ on the product $\mathbf{G}\times\mathbf{G}$, that is:

$$\Phi(\alpha\circ\beta,\alpha\circ\beta')=\Phi(\beta,\beta')\,, \tag{10}$$

for all triples α,β,β' such that the compositions $\alpha\circ\beta$ and $\alpha\circ\beta'$ make sense. Then it is clear that there is a one-to-one correspondence between invariant strongly

[11]The σ-algebra of measurable sets is then the power set of $\mathbf{G}$, that is $\Sigma=\mathcal{P}(\mathbf{G})$.

positive decoherence functionals D and positive semidefinite functions φ on the groupoid $\mathbf{G}$, the correspondence given by the assignment $\varphi \mapsto \Phi$ given in Eq. (9). Notice that the converse of (9) is given by: $\Phi \mapsto \varphi$, with $\varphi(\alpha) = \Phi(1_y, \alpha)$ if $\alpha \colon x \to y$.

The previous discussion shows that we may study invariant decoherence functionals on discrete groupoids, hence quantum measures on them, by studying the corresponding positive semidefinite functions φ. On the other hand a natural way to study decoherence functionals (and almost any abstract object in mathematics) is by looking at their representations (see for instance [Gu12] where recent results on this direction are shown). The interesting remark here is that, as it happens, the theory of positive semidefinite functions on groupoids provides a natural way to construct representations of groupoids and simultaneously of decoherence functionals. Such theory extends naturally that of positive semidefinite functions on groups with an analogue of Naimark's reconstruction theorem that provides a natural representation for the decoherence functional associated to the function φ. We will devote the following paragraphs to develop the theory in the case of discrete groupoids we are working with.

2.2.2. *States and positive semidefinite functions on groupoids.* Given the groupoid $\mathbf{G} \rightrightarrows \Omega$, it was observed in [Ib18b] the fundamental role in the dynamical interpretation of the theory played by its algebra as it represents the C^*-algebra of observables of the theory. In this section, we are going to put the emphasis on the C^*-algebra $\mathbb{C}[\mathbf{G}]$ of the groupoid as the algebra of physical transitions of the theory.

As it was discussed in the introduction, Sect. 1.2, the algebra $\mathbb{C}[\mathbf{G}]$ can be defined as the algebra of finite linear combinations of elements in $\mathbf{G}$. We may consider the abstract C^*-algebra $C^*(\mathbf{G})$ generated by $\mathbb{C}[\mathbf{G}]$, but in our context, we will make the simplifying assumption that we consider the closure of $\mathbb{C}[\mathbf{G}]$ with respect the norm induced from its fundamental representation. In other words, consider the Hilbert space $L^2(\Omega)$ (if Ω is finite, $L^2(\Omega)$ is just the complex linear space generated by Ω with the inner product defined by declaring that the elements x of Ω form an orthonormal basis). The fundamental representation $\pi_0 \colon \mathbb{C}[\mathbf{G}] \to \mathcal{B}(L^2(\Omega))$, is given by:

$$(\pi_0(\mathbf{a})\psi)(x) = \sum_{\alpha \in \mathbf{G}_+(x)} a_\alpha \psi(t(\alpha)) \,. \tag{11}$$

Notice that $\pi_0(\mathbf{1}) = I$ and $\pi_0(\mathbf{a}^*) = \pi_0(\mathbf{a})^\dagger$. Then we consider the norm $||\mathbf{a}|| = ||\pi_0(\mathbf{a})||_2$. The completion of $\mathbb{C}[\mathbf{G}]$ with respect to this norm is a C^*-algebra that we will take as the definition of the algebra of virtual transitions of the system, that we will denote in what follows as $\mathcal{A}_{\mathbf{G}}$.

Given a unital C^*-algebra a state ρ on it is a normalised positive linear functional, in our situation a state would be a linear map $\rho \colon \mathcal{A}_{\mathbf{G}} \to \mathbb{C}$ such that $\rho(\mathbf{1}) = 1$ and $\rho(\mathbf{a}^* \cdot \mathbf{a}) \geq 0$ for all $\mathbf{a}$. States play a particularly relevant role in the study of C^*-algebras. They form a convex domain in the dual space of the algebra denoted by $\mathcal{S}(\mathcal{A}_{\mathbf{G}})$ (or just $\mathcal{S}$ for short) and it is well-known that the structure of the algebra can be recovered from them. In the discussion to follow states are going to play an instrumental role because of the GNS construction and of the following observation: there is a one-to-one correspondence between states and continuous positive semidefinite functions φ on $\mathbf{G}$. The correspondence is as follows. Let $\varphi\colon \mathbf{G} \to \mathbb{C}$ be a positive semidefinite function, then we define the linear map $\rho_\varphi\colon \mathcal{A}_{\mathbf{G}} \to \mathbb{C}$ as (we consider for simplicity that Ω is finite[12]):

$$\rho_\varphi(\mathbf{a}) = \frac{1}{|\Omega|} \sum_\alpha a_\alpha \, \varphi(\alpha) \, .$$

Notice that ρ_φ is continuous with respect to the norm $|| \cdot ||$ and can be extended to all $\mathcal{A}_{\mathbf{G}}$. On the other hand clearly $\rho_\varphi(\mathbf{1}) = 1$ and a simple computation shows that:

$$\rho_\varphi(\mathbf{a}^* \cdot \mathbf{a}) = \sum_{(\alpha^{-1},\beta)\in\mathbf{G}_2} \bar{a}_\alpha a_\beta \, \varphi(\alpha^{-1} \circ \beta) \geq 0 \, , \tag{12}$$

by the very definition of φ. Conversely, given a state ρ on $\mathcal{A}_{\mathbf{G}}$ we define the function φ on $\mathbf{G}$ by restriction of ρ, that is:

$$\varphi_\rho(\alpha) = \rho(\alpha) \, ,$$

and then clearly φ_ρ is positive semidefinite because Eq. (12) can be read backwards. In such case we will say that φ_ρ is the characteristic function of the state ρ.

Then we conclude this section by realising that states on the algebra of generalised transitions of the system are associated with positive semidefinite functions on the groupoid, hence they determine invariant decoherence functionals and in consequence invariant quantum measures on $\mathbf{G}$, thus, in the case of finite groupoids, there is a one-to-one correspondence between states and invariant quantum measures on the groupoid. Notice that in such case, if $A \subset \mathbf{G}$, we get:

$$\begin{aligned} \mu_\rho(A) &= D(A,A) = \sum_{\alpha,\beta\in A} \Phi(\alpha,\beta) = \sum_{\alpha,\beta\in A} \delta(t(\alpha),t(\beta)) \, \varphi(\alpha^{-1}\circ\beta) \\ &= \sum_{\alpha,\beta\in A} \delta(t(\alpha),t(\beta)) \, \rho(\alpha^* \cdot \beta) \, . \end{aligned} \tag{13}$$

The remarkable formula (13) embodies in the abstract groupoid formalism Sorkin's quantum measure expression for systems described on spaces of histories[13] (see for instance [So16, eq. 14]) and explains the quadratic dependence of quantum measures on physical transitions.

[12] In the continuous of infinite case, we will assume that Ω carries a probability measure ν, the one used to define $L^2(\Omega,\nu)$ and then $|\Omega| = 1$.

[13] It is also remarkable that the delta function can be dropped in the last expression from (13) because if α^{-1} and β are not composable, then $\alpha^* \cdot \beta = 0$.

Notice that in the context developed in this section the evaluation of the state ρ on a transition α can be thought as the complex amplitude of the physical transition defined by α, thus the previous formula encodes the rule that 'probabilities' are obtained by module square of amplitudes in the abstract setting of groupoids. However in spite of all this, the previous expression for the quantum measure (and the decoherence functional) is given in abstract terms. We would like to describe them in terms of a concrete realization of the theory on a Hilbert space. This will be the task of the following sections.

3. Representations of decoherence functionals and quantum measures

3.1. **Representations of groupoids and algebras.** The background to construct representations of decoherence functionals on Hilbert spaces in the groupoid formalism of quantum mechanics will be provided by the representations of the groupoid $\mathbf{G} \rightrightarrows \Omega$ itself. Even if a functorial definition of representations of groupoids could be used, in the setting described in the previous sections, it is simpler to define a representation of the groupoid $\mathbf{G}$ as a representation of the C^*-algebra $\mathcal{A}_{\mathbf{G}}$ on the C^*-algebra $\mathcal{B}(\mathcal{H})$ of bounded operators on a complex separable Hilbert space $\mathcal{H}$, that is, we consider a C^*-algebra homomorphism $\pi \colon \mathcal{A}_{\mathbf{G}} \to \mathcal{B}(\mathcal{H})$ which is continuous in the sense that for any $\psi \in \mathcal{H}$, the map $\mathbf{a} \to ||\pi(\mathbf{a})\psi||$ is continuous. Notice that $\pi(\mathbf{1}) = I$ and $\pi(\mathbf{a}^*) = \pi(\mathbf{a})^\dagger$. In particular the fundamental representation π_0 discussed before, Eq. (11), is an example of an irreducible representation of the groupoid $\mathbf{G}$.

The theory of representations of groupoids shares many aspects with the theory of representations of groups (at least in the finite case this relation is well developed, see [Ib18d] for a description of the theory). We will not pretend to start such general discussion here. In what follows we will proceed by departing from a given state to construct explicit representations of the groupoid by means of the so called GNS construction.

Before describing this idea we would like to point out that if π is a nondegenerate representation[14] of the groupoid algebra $\mathcal{A}_{\mathbf{G}}$ on the Hilbert space $\mathcal{H}$, and ψ is a cyclic vector for such representation, that is the family of vectors $\pi(\mathbf{a})\psi$ span $\mathcal{H}$, then we may define the positive semidefinite function:

$$\varphi_{\pi,\psi}(\alpha) = \langle \psi, \pi(\alpha)\psi \rangle \,, \tag{14}$$

associated to the representation π and the cyclic vector ψ.

Notice that (14) actually defines a positive semidefinite function on $\mathbf{G}$ as it is shown by the following simple computation (as usual the sums are taken over all

[14] That is, a representation such that $\overline{\mathrm{span}\{\pi(\mathbf{a})\psi \mid \mathbf{a} \in \mathcal{A}_{\mathbf{G}}, \psi \in \mathcal{H}\}} = \mathcal{H}$.

composable pairs α_i^{-1}, α_j):

$$\begin{aligned}\sum_{i,j=1}^{n} \bar{\xi}_i \xi_j \varphi_{\pi,\psi}(\alpha_i^{-1} \circ \alpha_j) &= \sum_{i,j=1}^{n} \bar{\xi}_i \xi_j \langle \psi, \pi(\alpha_i^{-1} \circ \alpha_j)\psi\rangle = \sum_{i,j=1}^{n} \bar{\xi}_i \xi_j \langle \psi, \pi(\alpha_i)^\dagger \pi(\alpha_j)\psi\rangle \\ &= \sum_{i,j=1}^{n} \bar{\xi}_i \xi_j \langle \pi(\alpha_i)\psi, \pi(\alpha_j)\psi\rangle \leq \langle \sum_{i=1}^{n} \xi_i \pi(\alpha_i)\psi, \sum_{j=1}^{n} \xi_j \pi(\alpha_j)\psi\rangle \\ &= || \sum_{j=1}^{n} \xi_j \pi(\alpha_j)\psi ||^2 \geq 0 \,.\end{aligned}$$

and then, the state defined by $\varphi_{\pi,\psi}$ determines a quantum measure $\mu_{\pi,\psi}$ given as:

$$\begin{aligned}\mu_{\pi,\psi}(A) &= D_{\pi,\psi}(A,A) = \sum_{\alpha,\beta\in A} \delta(t(\alpha), t(\beta)) \varphi_{\pi,\psi}(\alpha^{-1} \circ \beta) \\ &= \sum_{\alpha,\beta\in A} \delta(t(\alpha), t(\beta)) \, \langle \pi(\alpha)\psi, \pi(\beta)\psi\rangle \,. \end{aligned} \tag{15}$$

In other words we may define a vector value measure $\nu_\pi \colon \Sigma \to \mathcal{H}$ as:

$$\nu_\pi(A) = \sum_{\alpha\in A} \pi(\alpha)\psi \,,$$

that represents the decoherence functional $D_{\pi,\psi}$ associated to the quantum measure $\mu_{\pi,\psi}$ (see [Do10b], [Gu12] for an account of the general theory). Notice finally, that the cyclic vector ψ for the representation π defines a state $\rho_{\pi,\psi}(\mathbf{a}) = \sum_\alpha a_\alpha \langle \psi, \pi(\alpha)\psi\rangle$ whose associated quantum measure is exactly that defined in Eq. (15).

It should also pointed out that the characteristic function $\varphi_{\pi,\psi}$ can also be expressed as:

$$\varphi_{\pi,\psi}(\alpha) = \mathrm{Tr}\,(\hat{\rho}_\psi \pi(\alpha)) \,,$$

where $\hat{\rho}_\psi$ denotes the rank-one orthogonal projector $|\psi\rangle\langle\psi|$ on $\mathcal{H}$ onto the one-dimensional space spanned by the vector $|\psi\rangle$. Then, if we consider instead the trivial projector defined by the identity operator I, we will get:

$$\varphi_{\pi,I}(\alpha) = \mathrm{Tr}\,(\pi(\alpha)) = \chi(\alpha) \,,$$

where the function $\chi = \varphi_{\pi,I}$ is commonly known as the character of the representation π. It is because of this that we would like to call the positive semidefinite function $\varphi_{\pi,\psi}$ defined by the state ψ, the *smeared* character of the representation π with respect to the state ψ.

3.2. **The GNS construction. Representations associated to states.** Because quantum measures μ, or for that matter, decoherence functionals, are associated to states ρ on the algebra of transitions of the system, it is just natural to use that state to construct a specific representation of it. The mechanism of constructing a representation given a state is well-known and goes under the name of the GNS construction and we will succinctly review it here in the present context.

Thus we will consider a state ρ on $\mathcal{A}_{\mathbf{G}}$. There is a canonical Hilbert space $\mathcal{H}_\rho$ associated to it defined as the completion of the quotient linear space $\mathcal{A}_{\mathbf{G}}/\mathcal{J}_\rho$, where $\mathcal{J}_\rho = \{\mathbf{a} \mid \rho(\mathbf{a}^* \cdot \mathbf{a}) = 0\}$ denotes the Gelfand ideal of ρ, with respect to the norm $||\cdot||_\rho$ associated to the state ρ and defined by:

$$||[\mathbf{a}]||_\rho = \rho(\mathbf{a}^* \cdot \mathbf{a}) \,, \qquad [\mathbf{a}] = \mathbf{a} + \mathcal{J}_\rho \in \mathcal{A}_{\mathbf{G}}/\mathcal{J}_\rho \,.$$

Thus the Hilbert space defined as $\mathcal{H}_\rho = \overline{\mathcal{A}_{\mathbf{G}}/\mathcal{J}_\rho}^{||\cdot||_\rho}$ will be called the GNS Hilbert space associated to the state ρ[15]. The parallelogram identity implies that the inner product $\langle \cdot, \cdot \rangle_\rho$ on $\mathcal{H}_\rho$ is given as:

$$\langle [\mathbf{a}], [\mathbf{b}] \rangle_\rho = \rho(\mathbf{a}^* \cdot \mathbf{b}) \,. \tag{16}$$

For our purposes it is fundamental to observe that there is a natural representation π_ρ of the C^*-algebra $\mathcal{A}_{\mathbf{G}}$ on $\mathcal{H}_\rho$, defined as:

$$\pi_\rho(\mathbf{a})([\mathbf{b}]) = [\mathbf{a} \cdot \mathbf{b}] \,,$$

for all $\mathbf{a} \in \mathcal{A}_{\mathbf{G}}$ and $[\mathbf{b}] \in \mathcal{H}_\rho$. Then clearly, the unit $\mathbf{1}$ of the algebra $\mathcal{A}_{\mathbf{G}}$ is mapped into the identity operator I and $\pi_\rho(\mathbf{a}^*) = \pi_\rho(\mathbf{a})^\dagger$.

The representation π_ρ is non-degenerate and the unit element $\mathbf{1}$ provides a cyclic vector for it. Denoting as customary by $|0\rangle$ the vector $[\mathbf{1}] \in \mathcal{H}_\rho$, it is clear that the set of vectors $\pi_\rho(\mathbf{a})|0\rangle$ span densely the vector space $\mathcal{H}_\rho$. The vector $|0\rangle$ is called (context depending) the *ground*, *vacuum* or *fundamental* vector of the GNS Hilbert space $\mathcal{H}_\varphi$. Notice that clearly $\langle 0 \mid 0 \rangle = \rho(\mathbf{1}^* \cdot \mathbf{1}) = 1$.

3.3. **Representation of decoherence functionals.** We shall consider now the state ρ associated to a given invariant decoherence functional D. In other words, according to the discussion in Sect. 2.2.2, we may consider a continuous positive semidefinite function φ on the groupoid $\mathbf{G}$ and the state ρ_φ (and the corresponding decoherence functional) associated to it (recall the fundamental equation relating all these notions, Eq. (13)). Denoting the GNS Hilbert space associated to the state ρ_φ by $\mathcal{H}_\varphi$, we get that $\mathcal{H}_\varphi$ is the completion of $\mathcal{A}_{\mathbf{G}}/\mathcal{J}_\varphi$, where $\mathcal{J}_\varphi$ denotes now the Gelfand's ideal:

$$\mathcal{J}_\varphi = \{\mathbf{a} \mid \sum_\alpha \bar{a}_\alpha a_\beta \, \varphi(\alpha^{-1} \cdot \beta) = 0\} \,,$$

with respect to the norm:

$$||[\mathbf{a}]||_\varphi^2 = \sum_\alpha \bar{a}_\alpha a_\beta \, \varphi(\alpha^{-1} \cdot \beta) \,,$$

[15]Such Hilbert space has been recognized in a closely related context by Dowker and Sorkin on its histories interpretation of quantum measures under the name of the 'history Hilbert space' [Do10].

that defines the inner product in $\mathcal{H}_\varphi$:

$$\langle [\mathbf{a}], [\mathbf{b}] \rangle_\varphi = \rho_\varphi(\mathbf{a}^* \cdot \mathbf{b}) = \sum_{\alpha,\beta} \bar{a}_\alpha b_\beta \varphi(\alpha^{-1} \cdot \beta)\,. \tag{17}$$

The fundamental vector $|0\rangle$ allows us to write the amplitude $\varphi(\mathbf{a})$ in the suggestive way:

$$\varphi(\mathbf{a}) = \rho_\varphi(\mathbf{a}) = \rho_\varphi(\mathbf{1}^* \cdot \mathbf{a}) = \langle 0 \mid [\mathbf{a}] \rangle_\varphi = \langle 0 \mid \pi_\varphi(\mathbf{a}) \mid 0 \rangle_\varphi\,, \tag{18}$$

where we have used (17) and the canonical representation $\pi_\varphi(\mathbf{a})|0\rangle = [\mathbf{a}]$.

In the same spirit as Eq. (18), the canonical representation of the algebra of transitions $\mathcal{A}_{\mathbf{G}}$ provided by the positive semidefinite function φ, allows to provide a representation of the decoherence functional in terms of amplitudes in the Hilbert space $\mathcal{H}_\varphi$ (and it constitutes also the particular instance of Eq. (15)):

$$D_\varphi(\alpha, \beta) = \varphi(\alpha^{-1} \cdot \beta) = \langle 0 \mid \pi_\varphi(\alpha)^\dagger \pi_\varphi(\beta) \mid 0 \rangle_\varphi\,.$$

Notice that if $t(\alpha) \neq t(\beta)$, then α^{-1} and β are not composable and $\alpha^{-1} \cdot \beta = 0$, hence $\langle [\alpha] \mid [\beta] \rangle_\varphi = 0$ or, equivalently $D_\varphi(\alpha, \beta) = 0$ (we will also say, mimicking the histories based approach to quantum mechanics, that the two transitions are decoherent).

Finally, notice that the quantum measure determined by the state ρ_φ has the definite expression:

$$\mu_\varphi(\{\alpha\}) = D_\varphi(\alpha, \alpha) = ||\pi_\varphi(\alpha)|0\rangle||_\varphi^2\,,$$

that shows it as the module square of an amplitude (even if the non-additivity of the quantum measure makes that its computation on subsets has to be performed according to the superposition rule provided by Eq. (15)).

3.4. **Naimark's reconstruction theorem for groupoids.** The discussion in the previous section can be summarised in the form of a theorem as:

Theorem 1. *Let $\mathbf{G} \rightrightarrows \Omega$ be a discrete groupoid with finite space Ω, then for any positive semidefinite function φ on $\mathbf{G}$ there exists a Hilbert space $\mathcal{H}$, a unitary representation π of the groupoid $\mathbf{G}$ on $\mathcal{H}$ and a vector $|0\rangle$ such that:*

$$\varphi(\alpha) = \langle 0|\pi(\alpha)|0\rangle\,.$$

In other words, any positive semidefinite function φ on a groupoid is the smeared character of a representation of the groupoid.

This statement can be considered as the extension of Naimark's reconstruction theorem for groupoids (admittedly, the rather particular instance of discrete groupoids with finite space of events). The 'reconstruction' word in theorem is justified from the following considerations.

Let π be a unitary representation of the groupoid $\mathbf{G}$ on the Hilbert space $\mathcal{H}$ (by that we mean that π defines a C^* representation of the C^*-algebra of the groupoid $\mathbf{G}$ on the C^*-algebra of bounded operators on the Hilbert space $\mathcal{H}$). Consider now a state ρ of the C^*-algebra $\mathcal{B}(\mathcal{H})$, that because of Gleason's theorem such state can be identified with a normalised Hermitean nonnegative operator $\hat{\rho}$. Then we define the function:

$$\varphi_\rho(\alpha) = \mathrm{Tr}\,(\hat{\rho}\,\pi(\alpha))\,. \tag{19}$$

It is immediate to check that φ_ρ defines a positive semidefinite function on $\mathbf{G}$. Then Thm. 1 shows that there exists a Hilbert space $\mathcal{H}'$, a representation π' and a state $\rho' = \mid 0\rangle\langle 0 \mid$, such that:

$$\varphi_\rho(\alpha) = \mathrm{Tr}\,(\rho'\,\pi'(\alpha))\,, \tag{20}$$

however, in principle, both representations of the function φ provided by Eqs. (19) and (20) are not equivalent. In the particular instance of groups, there is a positive answer to the previous question when the representation π is irreducible. In the more general situation of groupoids we are dealing here, we will not try to pursue these issues further that will be properly discussed elsewhere.

4. Local states and decoherence functionals

The general discussion brought on Sect. 3 has provided a general framework for a statistical interpretation of a groupoids based quantum theory by the hand of quantum measures and their realization by means of states on the algebra of virtual transitions of the theory, however no specific properties of the states have been called out that will reflect relevant physical properties of the system.

In this section we will discuss first the class of states (or quantum measures) that satisfy Feynman's composition of amplitudes law (1) and a particular instance of them, that will be called local states, that will strongly suggest a sum-over-histories interpretation of the corresponding quantum measure. In the remaining of this section, in order to simplify the presentation, we will restrict ourselves to the case of finite groupoids (even if the formalism extends naturally to countable discrete or even continuous groupoids).

4.1. **Reproducing states.**

States are just normalised positive linear functionals on the C^*-algebra of the groupoid, hence they are blind to the specific details of the algebraic structure of the algebra (they just preserve the positive cone of the algebra). It is true though that the C^*-algebra structure can recovered from the space of states, more precisely, because of Kadison's theorem [Ka51], the real part of a C^*-algebra is isometrically isomorphic to the space of all w^*-continuous affine functions on its state space, and then, by Falceto *et al* theorem, the C^*-algebra can be constructed on the space of affine function on the state space iff such space has the structure of a Lie-Jordan-Banach algebra [Fa13].

However, in general, the amplitudes $\varphi(\alpha)$ associated to a given state (or quantum measure) do not satisfy the reproducing property characteristic of Feynman's sum-over-histories interpretation of quantum mechanics. It is not hard to characterise a class of states such that the reproducing formula, that can also be called the abstract Chapman-Kolmogorov equation, given by Eq. (1), holds.

The reproducing condition for the state will be formulated in terms of the corresponding positive semidefinite function φ associated to it. Because $\varphi\colon \mathbf{G} \to \mathbb{C}$ is a function defined on the groupoid it is convenient to describe first the structure of the algebra $\mathcal{F}(\mathbf{G})$ of functions on the groupoid. In the case of finite groupoids, such algebra can be identified with the algebra of virtual transitions $\mathbb{C}[\mathbf{G}]$ (because of the existence of a natural basis on both of them). In any case the associative product $\star$ in $\mathcal{F}(\mathbf{G})$ is the natural one induced from the groupoid composition law, is called the convolution product, and is defined by the standard formula:

$$(f \star g)(\gamma) = \sum_{\substack{(\alpha,\beta)\in \mathbf{G}_2 \\ \alpha\circ\beta=\gamma}} f(\alpha)g(\beta)\,, \qquad f, g \in \mathcal{F}(\mathbf{G})\,, \gamma \in \mathbf{G}\,.$$

As in the case of the algebra of transitions $\mathcal{A}_{\mathbf{G}}$, if the space of events Ω is finite, there is a natural unit element, denoted again by $\mathbf{1}$ and defined as $\sum_{x\in\Omega}\delta_x$, with δ_x the function that takes the value 1 at 1_x and zero otherwise. In addition to the associative structure there is also an antiunitary involution operator $(\cdot)^*$, given by $f^*(\alpha) = \overline{f(\alpha^{-1})}$. The $*$-algebra $\mathcal{F}(\mathbf{G})$, like the algebra $\mathbb{C}[\mathbf{G}]$, has a natural representation on the space of square integrable functions on Ω that will be denoted with the same symbol π_0 and called again its fundamental representation. Such representation allows to define a norm on $\mathcal{F}(\mathbf{G})$ as $||f|| = ||\pi_0(f)||_2$, and equipped with the previous structures the algebra $\mathcal{F}(\mathbf{G})$ becomes a C^*-algebra[16].

We will say that a positive semidefinite function φ has the reproducing property if it satisfies

$$\varphi \star \varphi = \varphi \,, \tag{21}$$

or, in other words, φ is an idempotent element in $\mathcal{F}(\mathbf{G})$[17]. Finally given a positive semidefinite function φ, and given two events $a, a' \in \Omega$, will define the transition amplitude $\varphi_{a'a}$ as the sum of the amplitudes $\varphi(\alpha)$ for all transitions $\alpha \colon a \to a'$, in other words, we may think of $\varphi_{a'a}$ as the amplitude assigned to the transition of the event a to the event a' by the quantum measure μ_φ associated to the positive semidefinite function φ. Then it is easy to show that:

Proposition 2. *Let φ be an idempotent positive semidefinite function on the (finite) groupoid $\mathbf{G}$, that is, it satisfies the reproducing property condition Eq. (21), then the transition amplitudes $\varphi_{a'a}$ associated to it satisfy Feynman's amplitudes composition law (or the abstract Chapman-Kolmogorov reproducing equation):*

$$\varphi_{a'a} = \sum_{a''\in\Omega} \varphi_{a'a''}\varphi_{a''a} \,, \qquad \forall a, a' \in \Omega \,.$$

Proof. Clearly we get:

$$\varphi_{a'a} = \sum_{\gamma\in\mathbf{G}(a,a')} \varphi(\gamma) = \sum_{\gamma\in\mathbf{G}(a,a')} (\varphi\star\varphi)(\gamma) = \sum_{\gamma\in\mathbf{G}(a,a')} \sum_{\substack{(\alpha,\beta)\,\in\,\mathbf{G}_2 \\ \alpha\circ\beta=\gamma}} \varphi(\alpha)\varphi(\beta) \,, \tag{22}$$

where we have used the reproducing property (21) for φ and the definition of the convolution product. But now, if $\gamma = \alpha\circ\beta$, because $\gamma \colon a \to a'$, then $\beta \colon a \to a''$ and $\alpha \colon a'' \to a'$ for some $a'' \in \Omega$, then the last term in the r.h.s. of Eq. (22) can be written as:

$$\sum_{\gamma\in\mathbf{G}(a,a')} \sum_{\substack{(\alpha,\beta)\,\in\,\mathbf{G}_2 \\ \alpha\circ\beta=\gamma}} \varphi(\alpha)\varphi(\beta) = \sum_{a''\in\Omega} \sum_{\alpha\in\mathbf{G}(a'',a')} \sum_{\beta\in\mathbf{G}(a,a'')} \varphi(\alpha)\varphi(\beta) = \sum_{a''\in\Omega} \varphi_{a'a''}\varphi_{a''a} \,.$$

□

[16]See our previous paper [Ib18b] for the interpretation of the C^*-algebra $\mathcal{F}(\mathbf{G})$ as the algebra of observables of the system and [Re80] or [La98] for a detailed construction of the C^*-algebra of functions on a continuous or Lie groupoid.

[17]Idempotent elements in algebras play a prominent role, for instance characters of irreducible representations are particular instances of them.

We should point out that the transition amplitude $\varphi_{a'a}$ can also be expressed as the transition amplitude associated to the representation of the function φ in the space $\mathcal{H}_\Omega$, that is (see [Ib18b]):

$$\varphi_{a'a} = \langle a'|\pi_0(\varphi)|a\rangle = \sum_{\gamma\in\mathbf{G}(a,a')} \varphi(\gamma)\,.$$

Then, a simple alternative proof of the previous proposition is obtained by the following computation:

$$\langle a'|\pi_0(\varphi)|a\rangle = \langle a'|\pi_0(\varphi\star\varphi)|a\rangle = \langle a'|\pi_0(\varphi)\pi_0(\varphi)|a\rangle = \sum_{a''\in\Omega}\langle a'|\pi_0(\varphi)|a''\rangle\langle a''|\pi_0(\varphi)|a\rangle\,,$$

where we have used the fact that π_0 is a representation of the C^*-algebra $\mathcal{F}(\mathbf{G})$. Then $\pi_0(\varphi\star\varphi) = \pi_0(\varphi)\pi_0(\varphi)$ and the projectors $\pi_0(1_a) = |a\rangle\langle a|$ provide a resolution of the identity in $\mathcal{H}_\Omega$.

4.2. **Local states.** Generic states are insensitive to the 'local' structure of the algebra of transitions codified by the composition law $\alpha\circ\beta$, that is, the amplitudes $\varphi(\alpha\circ\beta)$ are, in general, not directly related to the amplitudes of the factors $\varphi(\alpha)$ and $\varphi(\beta)$.

Thus there is a natural class of states determined as those that can be constructed out of the information provided by the factors, that is, states that are characterised in terms of the values of the associated smeared character φ on a family of transitions generating the groupoid. Then we will say that a state, or the corresponding smeared character φ, is local if for any pair of composable transitions $(\alpha,\beta)\in\mathbf{G}_2$:

(23) $$\varphi(\alpha\circ\beta) = \varphi(\alpha)\varphi(\beta)\,.$$

The reversibility of transitions suggest the unitarity preserving property:

(24) $$\varphi(\alpha^{-1}) = \varphi(\alpha)^*\,,$$

that will be consider too in addition to the strict locality property (23). Condition (24) is independent of the locality condition (23) and it can be lifted when dealing with open systems. Notice that as a consequence of the previous definition, Eq. (23), $\varphi(1_x) = 1$ (because $\varphi(1_x\circ 1_x) = \varphi(1_x)$) and $|\varphi(\alpha)| = 1$ (because of Eq. (24))[18].

Even if seems, Eq. (23), that the function $\varphi\colon \mathbf{G}\to\mathbb{C}$ defines a 'representation' of the groupoid, this is not so. A representation of the groupoid $\mathbf{G}$ is a functor R from $\mathbf{G}$ to the category of linear spaces **Vect**. Thus, to any event x we must associate a linear space $V_x = R(x)$. The simplest possibility would be to associate the 1-dimensional linear space $\mathbb{C}$ to each event $a\in\Omega$ (notice the $R(1_x)$ must be invertible, thus $R(x)\neq\{\mathbf{0}\}$). Thus the total space would have dimension equal

[18]The reader should notice that in this abstract construction, 'local' doesn't refer to any space-time property, however because of the consistency of the amplitude values $\varphi(\alpha)$ with the composition law, the structure of the space of events $\{a\}$ is preserved.

to the order of Ω. Hence, unless $|\Omega| = 1$, as it happens in the case of ordinary groups, the smallest possible representation of a groupoid has dimension larger than 1. Such smallest representation is obviously irreducible and is what we have been calling the fundamental representation π_0 of the groupoid.

On the other hand, in general, a local state ρ is neither a representation of the algebra of virtual transitions. If $\rho\colon \mathbb{C}[\mathbf{G}] \to \mathbb{C}$ is the state such that $\rho(\mathbf{a}) = \sum_\alpha a_\alpha\, \varphi(\alpha)$ with φ satisfying Eq. (23), then it is not true in general that $\rho(\mathbf{a}\cdot\mathbf{b})$ agrees with the product $\rho(\mathbf{a})\rho(\mathbf{b})$. The reason for this is that in the evaluation of $\rho(\mathbf{a}\cdot\mathbf{b})$ only the terms $\varphi(\alpha\circ\beta)$ with α and β composable will appear while in $\rho(\mathbf{a})\rho(\mathbf{b})$ all products $\varphi(\alpha)\varphi(\beta)$ will contribute, making the two of them different.

Notice that the amplitude $\varphi(\alpha)$ of a local state can always be written as:

$$\varphi(\alpha) = e^{is(\alpha)}\,, \tag{25}$$

for a real-valued function $s\colon \mathbf{G} \to \mathbb{R}$ satisfying the following properties:

$$s(1_x) = 0\,, \qquad \forall x \in \Omega\,, \tag{26}$$

$$s(\alpha\circ\beta) = s(\alpha) + s(\beta)\,, \tag{27}$$

for any pair of composable transitions α, β, and

$$s(\alpha^{-1}) = -s(\alpha)\,. \tag{28}$$

we will call a real valued function s on a groupoid satisfying the conditions (26), (27), (28), an action.

Even if the discussion of the statistical interpretation of the formalism has been done without reference to any particular dynamics, the structure of local states is strongly reminiscent of Dirac-Feynman definition of amplitudes in the standard space-time interpretation of quantum mechanics, that is why we will call such function s an action. We may ask now what properties must an action s possess beyond those expressed in the previous equations for the function $\varphi = e^{is}$ to define a state, that is, to be positive semidefinite, and if it is, when it will have the reproductive property.

Theorem 3. *Let $s\colon \mathbf{G} \to \mathbb{R}$ be an action on a finite groupoid $\mathbf{G}$. Then the function $\varphi = e^{is}$ is positive semidefinite and satisfies the reproductive property, $\varphi = \varphi\star\varphi$. We will call the state defined in this way the dynamical local state of the theory defined by the action s.*

Proof. Let $n \in \mathbb{N}$, $\xi_i \in \mathbb{C}$ and $\alpha_i \in \mathbf{G}$, $i = 1, \ldots, n$.

We will prove that e^{is} is positive semidefinite by induction, i.e., we will show that:

$$S_n = \sum_{i,j=1}^n \bar{\xi}_i \xi_j e^{is(\alpha_i^{-1}\circ\alpha_j)} \geq 0\,,$$

where only composable pairs $\alpha_i^{-1} \circ \alpha_j$ appear in the expansion of the sum (notice that if α_i^{-1} is composable with α_j, then α_j^{-1} is composable with α_i), by complete induction on n.

Thus, if $n = 1$, there is only a complex number ξ and α and the sum $S_1 = |\xi|^2 \geq 0$. If $n = 2$, we will be considering complex numbers ξ_1, ξ_2 and transitions α_1, $\alpha_2 \in \mathbf{G}$. There will be two possibilities, either α_1 and α_2 are composable or they are not. If they are, then we have:

$$S_2 = \sum_{i,j=1}^{2} \bar{\xi}_i \xi_j e^{is(\alpha_i^{-1} \circ \alpha_j)} = \sum_{i,j=1}^{2} \bar{\xi}_i \xi_j e^{-is(\alpha_i)} e^{is(\alpha_j)} = |\xi_1 e^{is(\alpha_1)} + \xi_2 e^{is(\alpha_2)}|^2 \geq 0 \,,$$

while if α_1^{-1} and α_2 are not composable then, $S_2 = |\xi_1|^2 + |\xi_2|^2 \geq 0$. To understand the general situation we may discuss the case $n = 3$ too. Then we will have three complex numbers ξ_1, ξ_2, ξ_3 and three transitions α_1, α_2 and α_3. There are three cases: all three transitions are composable, two are composable, say α_1, α_2 and one is not, and the three are not composable or disjoint. In the first case a simple computation shows that:

$$S_3 = |\xi_1 e^{is(\alpha_1)} + \xi_2 e^{is(\alpha_2)} + \xi_3 e^{is(\alpha_3)}|^2 \geq 0 \,,$$

while in the second and third, we get respectively:

$$S_3 = |\xi_1 e^{is(\alpha_1)} + \xi_2 e^{is(\alpha_2)}|^2 + |\xi_3|^2 \geq 0 \,, \quad S_3 = |\xi_1|^2 + |\xi_2|^2 + |\xi_3|^2 \geq 0 \,.$$

Let us consider n arbitrary, then the relation $i \sim j$ if α_i^{-1} is composable with α_j (or in other words, if the targets of α_i and α_j are the same, $t(\alpha_i) = t(\alpha_j)$) is an equivalence relation in the set of indices $I_n = \{1, 2, \ldots, n\}$. The set I_n is decomposed into equivalence classes $I_x = \{i_{x_1}, \ldots, i_{x_r}\}$, that will correspond to all transitions α_i such that $t(\alpha_i) = x$, and each class will have a number of elements $n_x \leq n$. Then if $n_x = n$, there is only one class, all pair of transitions α_i^{-1}, α_j are composable and then:

$$S_n = \Big| \sum_{k=1}^{n} \xi_k e^{is(\alpha_k)} \Big|^2 \geq 0 \,.$$

On the other hand if there is more than one equivalence class, then $n_x < n$ for all $x \in \Omega$, and then we have:

$$S_n = \sum_{x \in \Omega} \sum_{j_x, k_x \in I_x} \bar{\xi}_{j_x} \xi_{k_x} e^{-is(\alpha_{j_x})} e^{is(\alpha_{k_x})} = \sum_{x \in \Omega} \Big| \sum_{k_x \in I_x} \xi_{k_x} e^{is(\alpha_{k_x})} \Big|^2 \geq 0 \,,$$

where in the last step in the previous computation we have used the induction hypothesis. This shows that $\varphi = e^{is}$ is positive semidefinite.

To prove the reproducing property, we normalise the smeared character φ properly as:

$$\varphi(\alpha) = \frac{|\Omega|}{|\mathbf{G}|} e^{is(\alpha)} \,.$$

Then a simple computation shows that:

$$\begin{aligned}
\varphi \star \varphi(\gamma) &= \frac{|\Omega|^2}{|\mathbf{G}|^2} \sum_{\substack{(\alpha,\beta)\in \mathbf{G}_2 \\ \alpha\circ\beta=\gamma}} e^{is(\alpha)} e^{is(\beta)} \\
&= \frac{|\Omega|^2}{|\mathbf{G}|^2} \sum_{\substack{(\alpha,\beta)\in \mathbf{G}_2 \\ \alpha\circ\beta=\gamma}} e^{is(\alpha\circ\beta)} \\
&= \frac{|\Omega|}{|\mathbf{G}|} e^{is(\gamma)} = \varphi(\gamma)\,, \qquad (29)
\end{aligned}$$

where in the step (29) in the previous computation, we have used that the argument of the sum, $e^{is(\alpha\circ\beta)}$, is constant and equal to $e^{is(\gamma)}$ whenever $\alpha\circ\beta=\gamma$, but because the number of composable transitions α, β such that $\alpha\circ\beta=\gamma$ is exactly $|\mathbf{G}|/|\Omega|$, then we get the required factor and the conclusion.

Let us justify this last statment. First, notice that if $\gamma\colon x\to y$, then for any $\alpha\colon x\to z$ there is exactly one $\beta=\alpha^{-1}\circ\gamma$ such that $\alpha\circ\beta=\gamma$, then the number of pair transitions factorising $\gamma\colon x\to y$ is $|\mathbf{G}_+(x)|$, but $\sqcup_{x\in\Omega}\mathbf{G}_+(x)=\mathbf{G}$, then $|\mathbf{G}|=|\Omega||\mathbf{G}_+(x)|$ and the statement is proved. □

We can summarise all previous discussion by saying that we can understand the description of a quantum system in the groupoid formalism (which provides an abstraction of Schwinger algebra of measurements) as a grade-2 measure theory provided by an invariant quantum measure μ. Such quantum measure is characterised by a positive semidefinite function φ on the groupoid and for any action function s on the groupoid, the function $\varphi=e^{is}$ is local, satisfies the reproducing property and defines uniquely a quantum measure μ_s whose decoherence functional D_s has Sorkin's form:

$$D_s(\alpha,\beta)=e^{-iS(\alpha)}e^{iS(\beta)}\delta(t(\alpha),t(\beta))\,.$$

Here α and β denote two transitions in the groupoid $\mathbf{G}$ and we have made explicit the delta function of the targets.

5. The statistical interpretation of Schwinger's transformation functions

In the previous sections we have been discussing how the notion of state on the algebra of transitions of a quantum system described by a groupoid $\mathbf{G} \rightrightarrows \Omega$ provides a statistical interpretation of the theory connecting it directly with Sorkin's notion of quantum measure and the theory of decoherence functionals.

In this section, and as anticipated in the introduction, we will provide a natural statistical interpretation of Schwinger's transformation functions by relying again on the key notion of states. This time we will provide a natural interpretation of transition amplitudes on the fundamental representation of a given groupoid by using particularly simple states. Moreover a judiciously use of the the fundamental invariance of the description of the system with respect to changes of systems of observables will provide the desired interpretation.

5.1. **Equivalence of algebras of observables.** It is a fundamental assumption of the theory developed so far that if we select a complete set of observables $\mathbf{A}$ for the system, the algebra of observables $\mathcal{A}$ of the system will be isomorphic to the C^*-algebra of observable $\mathcal{F}(\mathbf{G_A})$ of the goupoid $\mathbf{G_A}$ determined by the complete system $\mathbf{A}$ [Ib18b]. The groupoid $\mathbf{G_A}$ will consist of all transitions $\alpha\colon a \to a'$ among events $a \in \Omega_A$ determined by the set of compatible observables $\mathbf{A}$.

If $\mathbf{B}$ is another complete set of observables describing the given system, then the C^*-algebra $\mathcal{F}(\mathbf{G_B})$ associated to this description must be isomorphic to the algebra $\mathcal{A}$ of observables of the system too, that is, there must be an isomorphism

$$\tau_{AB}\colon \mathcal{F}(\mathbf{G_A}) \to \mathcal{F}(\mathbf{G_B})\,,$$

between the corresponding C^*-algebras describing the system in both reference systems $\mathbf{A}$ and $\mathbf{B}$ because the description of the system cannot depend on the choice of a particular set of compatible observables. This independence of the description with respect to the chosen 'reference frame' was stated as a 'relativity principle' in [Ib18b]. Notice that together with τ_{AB} there is an isomorphism $\tau_{BA}\colon \mathcal{F}(\mathbf{G_B}) \to \mathcal{F}(\mathbf{G_A})$ and then it is natural to conclude that $\tau_{AB} = \tau_{BA}^{-1}$. In the same vein if $\mathbf{C}$ is another complete system of observables yet, then there will exists isomorphisms of C^*-algebras $\tau_{BC}\colon \mathcal{F}(\mathbf{G_B}) \to \mathcal{F}(\mathbf{G_C})$ and $\tau_{AC}\colon \mathcal{F}(\mathbf{G_A}) \to \mathcal{F}(\mathbf{G_C})$, that will be assumed to satisfy the natural composition law[19]:

$$\tau_{BC} \circ \tau_{AB} = \tau_{AC}\,.$$

The isomorphism τ_{AB} induces by duality an isomorphism between the algebras of transitions of the system, that is, between the algebra $\mathbb{C}[\mathbf{G_A}]$ and $\mathbb{C}[\mathbf{G_B}]$ of

[19]This assumption corresponds to consider that the categorical notions behind the structures we are dealing with are defined in the strong sense, while there is always the possibility of using a weaker version of them where equalities are always defined up to isomorphisms in the corresponding categroy.

transitions as constructed from the systems of observables **A** and **B**. We will denote the induced isomorphism with the same symbol τ_{AB}, that is, we have:

$$(\tau_{AB} f)(\mathbf{a}) = f(\tau_{AB}(\mathbf{a}))\,,$$

where, as indicated in §5, $\mathbf{a} = \sum_\alpha a_\alpha \alpha$ is a virtual transition, and the generalised observable functions $f\colon \mathbf{G} \to \mathbb{C}$, extends by linearity to $\mathbb{C}[\mathbf{G}]$, that is, $f(\mathbf{a}) = \sum_\alpha a_\alpha f(\alpha)$.

5.2. **Transition amplitudes again.** Finally, let us recall that a real observable is a function $f \in \mathcal{F}(\mathbf{G}_\mathcal{A}) = \mathcal{A}$ such that $f^* = f$, that is, a self-adjoint element in the C^*-algebra $\mathcal{A}$. We will define the *transition amplitude* of the observable f between two events a and a' as the sum of the values of the observable over all transitions connecting a and a' and we will denote it by $\langle a; f; a' \rangle$:

$$\langle a; f; a' \rangle = \sum_{\alpha \in \mathbf{G}(a,a')} f(\alpha)\,.$$

Notice that

$$\langle a; f^*; a' \rangle = \sum_{\alpha \in \mathbf{G}(a,a')} f^*(\alpha) = \sum_{\alpha \in \mathbf{G}(a',a)} \bar{f}(\alpha) = \overline{\langle a'; f; a \rangle}\,,$$

and if we denote by $\langle a; a' \rangle$ the amplitude corresponding to the unit $\mathbf{1}$, that is, $\langle a; a' \rangle = \langle a; \mathbf{1}; a' \rangle$, then:

$$\langle a; a' \rangle = \delta(a, a')\,,$$

because

$$\langle a; a \rangle = \langle a; \mathbf{1}; a \rangle = \sum_{a' \in \Omega} \langle a; \delta_{a'}; a \rangle = \sum_{a' \in \Omega} \sum_{\alpha \in \mathbf{G}(a,a)} \delta_{a'}(\alpha) = 1\,,$$

and clearly $\langle a; a' \rangle = 0$, if $a \neq a'$, as $\mathbf{1}$ must be evaluated on transitions α with different source and target.

Another interesting observable is provided by the 'incidence matrix' observable $\mathbb{I} = \sum_{\alpha \in \mathbf{G}} \delta_\alpha$. Notice that $\mathbb{I}^* = \mathbb{I}$ and:

$$\langle a'; \mathbb{I}; a \rangle = \sum_{\alpha \in \mathbf{G}(a',a)} \mathbb{I}(\alpha) = |\mathbf{G}(a', a)|\,.$$

It is also relevant to point out the if φ is a positive definite function on $\mathbf{G}$ then $\varphi^* = \varphi$ (notice that because $\sum_{i,j=1}^n \bar{\xi}_i \xi_j \varphi(\alpha_i^{-1} \circ \alpha_j) \geq 0$ for all ξ_i, then $\overline{\varphi(\alpha_i^{-1} \circ \alpha_j)} = \varphi(\alpha_j^{-1} \circ \alpha_i)$ for all composable α_i^{-1}, α_j, but then it holds for all α), then

$$\langle a'; \varphi; a \rangle = \sum_{\alpha \in \mathbf{G}(a',a)} \varphi(\alpha) = \varphi_{a'a}\,,$$

and the transition amplitude $\langle a'; \varphi; a \rangle$ is just the transition amplitude of the state φ considered in Sect. 4.1.

5.3. **The states ρ_x and their associated GNS constructions.** To relate the definition of transition amplitudes with the standard interpretation of such functions in terms of vector-states and operators, and eventually with Schwinger's transformation functions, we have to select a representation of the theory.

As we discussed before, Sect. 3.2, the representations of the C^*-algebra $\mathbb{C}[\mathbf{G}]$ are defined via the GNS construction. Hence following the spirit so far, we will choose a particular state that will provide a particular representation of transition amplitudes. For that, and as a further illustration of the GNS construction, we will consider the simple state ρ_x defined by the function δ_x, that is $\rho_x(\mathbf{a}) = a_x$ where $\mathbf{a} = \sum_\alpha a_\alpha \, \alpha$, that is, ρ_x assigns to any virtual transition $\mathbf{a}$ the coefficient of the unit 1_x. Clearly $\rho_a(\mathbf{1}) = 1$ and

$$\rho_x(\mathbf{a}^* \cdot \mathbf{a}) = \sum_{\alpha \in \mathbf{G}_+(x)} |a_\alpha|^2 \geq 0 \, , \tag{30}$$

that shows that ρ_x is an state indeed.

Following the GNS construction described in Sect. 3.2 we see that the Hilbert space $\mathcal{H}_{\rho_x}$, denoted in what follows by $\mathcal{H}_x$, is the Hilbert space of functions Φ defined on $\mathbf{G}_+(x)$ with the standard inner product. In fact from Eq. (30) we see that the Gelfand ideal $\mathcal{J}_x = \{\mathbf{a} \mid \rho_x(\mathbf{a}^* \cdot \mathbf{a}) = 0\}$ consists of all $\mathbf{a}$ such that the coefficients of transitions $\alpha \in \mathbf{G}_+(x)$ vanish. That means that the cocient space $\mathbb{C}[\mathbf{G}]/\mathcal{J}_x$ can be identified with the space of transitions in $\mathbf{G}_+(x)$, thus given any $\mathbf{a} \in \mathbb{C}[\mathbf{G}]$ we will use the notation $\mathbf{a}_x$ for the restriction to $\mathbf{G}_+(x)$, i.e., $\mathbf{a}_x$ is obtained from $\mathbf{a}$ by putting to zero all coefficients a_α with $\alpha \notin \mathbf{G}_+(x)$ or, in other words $\mathbf{a}_x = \mathbf{a} \cdot 1_x$. Moreover the induced inner product $\langle \cdot, \cdot \rangle_x$ in $\mathcal{H}_x$ induced by ρ_x is given by, Eq. (16):

$$\langle \mathbf{a}_x, \mathbf{a}'_x \rangle_x = \rho_x(\mathbf{a}^* \cdot \mathbf{a}') = \sum_{\alpha \in \mathbf{G}_+(x)} \bar{a}_\alpha a'_\alpha \, . \tag{31}$$

In particular the unit $\mathbf{1}$ determines the fundamental vector $\mathbf{1}_x = 1_x \in \mathcal{H}_x$. The algebra $\mathbb{C}[\mathbf{G}]$ is represented in $\mathcal{H}_x$ as $\pi_x(\mathbf{a})\mathbf{a}'_x = (\mathbf{a} \cdot \mathbf{a}')_x = \mathbf{a} \cdot \mathbf{a}'_x$, and clearly 1_x is a cyclic vector for such representation. Now instead of denoting by $|0\rangle$ the ground vector of the representation π_x for convenience we will denote it by $|x\rangle$). Thus if $\mathbf{a}$ is a virtual transition, we have:

$$\pi_x(\mathbf{a})|x\rangle = \mathbf{a}_x \, ,$$

that, in order to have a homogeneous notation we can denote as $\mathbf{a}_x = |\mathbf{a}\rangle_x$ where the subscript x indicates that the vector $|\mathbf{a}\rangle_x$ belongs to the Hilbert space $\mathcal{H}_x$. Thus, using this notation in Eq. (31) we have:

$$\langle \mathbf{a}_x, \mathbf{a}'_x \rangle = \sum_{\alpha \in \mathbf{G}_+(x)} \bar{a}_\alpha a'_\alpha = \langle \mathbf{a} \mid \mathbf{a}' \rangle_x \, ,$$

which is the convenient form of expressing the inner product that we will follow. With this notation the amplitude defined by the state ρ_x on a virtual transition $\mathbf{a}$

can be written as, recall Eq. (18):

$$\rho_x(\mathbf{a}) = \langle x \mid \mathbf{a} \rangle_x \,. \tag{32}$$

5.4. **Transformation functions and transition amplitudes.** Now we are ready to interpret Schwinger's transformation functions $\langle b|a\rangle$ as transition amplitudes and hence to provide them with a proper statistical interpretation. Let us recall the according to Schwinger, the transformation function $\langle b|a\rangle$ "is a number characterising the statistical relation relation between the states b and a", and reflects the fact 'that only a determinate fraction of the systems emerging from the first stage will be accepted by the second stage".

In the formalism we have developed Schwinger's transformation function will be given by the isomorphism τ_{AB} that relates the $\mathbf{A}$ and $\mathbf{B}$ pictures of the system. Moreover this isomorphism should bobtained from the 2-groupoid structure of the theory (see [Ib18a]) but we will not dwell on this interpretation. Instead we would like to provide an statistical interpretation of the complex number $\langle b|a\rangle$ appearing in Schwinger's formalism as transition amplitudes. For that consider that in the description provided by the complete family $\mathbf{B}$ of observables, we want to understand the statistical relation between the outcome b, i.e., the transition 1_b in the algebra $\mathbb{C}[\mathbf{G_B}]$, and the transition 1_a corresponding to the outcome a with respect to the description provided by the family $\mathbf{A}$, that is, the algebra $\mathbb{C}[\mathbf{G_A}]$. Then such relation is provided by the amplitude of the state ρ_a defined by a in $\mathbb{C}[\mathbf{G_A}]$ as described in the previous section, Sect. 5.3 on the transition defined by 1_b that will be $\tau_{BA}(1_b) \in \mathbb{C}[\mathbf{G_A}]$. But then, using Eq. (32), we get:

$$\rho_a(\tau_{BA} 1_b) = \langle a | \tau_{BA}(1_b) \rangle_a \,.$$

If we denote the vector state in the Hilbert space $\mathcal{H}_a$ defined by the transition $\tau_{BA}(1_b)$ by $|b\rangle$, that is:

$$|b\rangle = \pi_a(\tau_{BA}(1_b))|a\rangle \,,$$

we get that the transition amplitude of the event b with respect to the state defined by a, that we may denote consistently as φ_{ab}, is given by:

$$\varphi_{ab} = \langle b|a\rangle \,.$$

Notice that the we could have proceed the other way around, exchanging the roles of a and b, then repeating the argument we get that the transition amplitude φ_{ba} of the event a with respect to the state defined by b, would have been:

$$\varphi_{ba} = \langle a|b\rangle = \overline{\langle b|a\rangle} = \overline{\varphi}_{ab} \,.$$

Notice that the previous identities follow from the duality of states and transitions and the properties of the isomorphisms τ_{AB}, that is:

$$\rho_a(\tau_{BA}(1_b)) = \rho_{\tau_{AB}(1_a)}(1_b) = \rho_b(\tau_{AB}(1_a)) \,, \qquad \forall a \in \Omega_A, b \in \Omega_B \,.$$

6. A simple application: the qubit and the two-slit experiment

6.1. **The qubit.** We can illustrate the ideas discussed along this paper by using the qubit system. The qubit system is the simplest nontrivial quantum system and in the groupoid formalism correspond to the groupoid defined by the graph A_2, that is the space $\Omega_2 = \{+, -\}$ consists of two events $+$, $-$, and there is one non-trivial transition $\alpha\colon - \to +$. In addition there are the units $1_\pm$ and the inverse $\alpha^{-1}\colon + \to -$, with $\alpha^{-1} \circ \alpha = 1_-$, $\alpha \circ \alpha^{-1} = 1_+$ (see Fig. 1). This scheme abstracts the simplest situation of a physical system evolving in time and producing two outcomes denoted by $+$ and $-$.

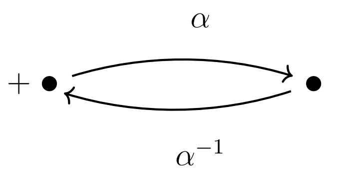

FIGURE 1. The abstract qubit, A_2.

The corresponding groupoid will be denoted by A_2 again and its algebra $\mathbb{C}[A_2] = \{\mathbf{a} = a_+ 1_+ + a_- 1_- + a_\alpha \alpha + a_{\alpha^{-1}} \alpha^{-1}\}$ is easily seen to be isomorphic to the algebra $M_2(\mathbb{C})$ of 2×2 complex matrices. The identification provided by the assignments:

$$
\begin{aligned}
1_+ &\mapsto \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad 1_- \mapsto \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\
\alpha &\mapsto \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \alpha^{-1} \mapsto \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.
\end{aligned}
$$

Then a virtual transition $\mathbf{a}$ is associated to the matrix:

$$
A = \begin{bmatrix} a_+ & a_\alpha \\ a_{\alpha^{-1}} & a_- \end{bmatrix},
$$

and a^* is associated to the matrix $A^\dagger$. The C^* norm $|| \cdot ||$ is just the matrix operator norm and the fundamental representation π_0 of the algebra becomes the natural defining representation of $M_2(\mathbb{C})$ on $\mathbb{C}^2$. The vectors associated to the unit elements $1_\pm$ are given by:

$$
|+\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad |-\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix},
$$

and thus an arbitrary vector in $\mathcal{H}_2 = \mathbb{C}^2$ is written as $|\psi\rangle = \psi_+|+\rangle + \psi_-|-\rangle$.

The space of states of the groupoid algebra $\mathbb{C}[A_2]$ can be identified with the space of density operators on $\mathcal{H}_2$, that is normalized, non-negative, self-adjoint operators $\hat{\rho}$ on $\mathcal{H}_2$. Density operators can be parametrized as:

$$
\hat{\rho} = \frac{1}{2}(\mathbb{I} - \mathbf{r} \cdot \boldsymbol{\sigma})
$$

with $\mathbf{r} \in \mathbb{R}^3$ a vector in Bloch's sphere, $r = ||\mathbf{r}|| \leq 1$, and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, the standard Pauli sigma matrices.

According to Thm. 3, local states have the form $\varphi = e^{is}$, with s and action function. Then, let $s \colon A_2 \to \mathbb{R}$ given by:

$$s(1_\pm) = 0\,, \qquad s(\alpha) = -s(\alpha^{-1}) = S\,.$$

Clearly the function s defined in this way satisfies the additive property (27) and the state defined by $\rho_S(\mathbf{a}) = \sum_{i,j} \bar{a}_i a_j \varphi(\alpha_i^{-1} \circ \alpha_j)$ is a local (and reproductive) state. The characteristic function φ_S defined by the action s is given by:

$$\varphi_s(1_\pm) = 1\,, \qquad \varphi_s(\alpha) = \overline{\varphi_s(\alpha^{-1})} = e^{-iS}\,,$$

and the associated state:

$$\hat{\rho}_S = \frac{1}{2}\begin{bmatrix} 1 & e^{-iS} \\ e^{iS} & 1 \end{bmatrix}.$$

Notice that $\hat{\rho}_S\hat{\rho}_S = \hat{\rho}_S$, thus it satisfies the reproducing property (it can also be checked directly that $\varphi_S \star \varphi_S = \varphi_S$).

The decoherence functional defined by this state is given by the 4×4 matrix D_S whose entries (i,j) correspond to the values $D_S(\alpha_i, \alpha_j) = \frac{1}{4}\varphi_S(\alpha_i^{-1} \circ \alpha_j)\delta(t(\alpha_i), t(\alpha_j))$, with α_i running through the list $1_+, 1_-, \alpha, \alpha^{-1}$, thus for instance $D(1,1) = D_S(1_+, 1_+) = \frac{1}{2}\varphi_S(1_+^{-1} \circ 1_+) = 1/4$, $D(1,2) = D_S(1_+, 1_-) = \frac{1}{2}\varphi_S(1_+^{-1} \circ 1_-) = 0$, and so on, thus we get:

$$D_S = \frac{1}{4}\begin{bmatrix} 1 & 0 & e^{-iS} & 0 \\ 0 & 1 & 0 & e^{iS} \\ e^{iS} & 0 & 1 & 0 \\ 0 & e^{-iS} & 0 & 1 \end{bmatrix}.$$

As it was discussed in the main text, the decoherence functional describes the structure of the quantum measure μ_S, and hence the statistical interpretation, associated to the system A_2 in the state ρ_S.

6.2. **The double slit experiment.** In order to understand better some of the implications of the previous discussion it is revealing to compare the qubit system with the double slit experiment. For the purposes of the present paper, we will use the analysis of the double slit experiment carried on in [Ga09] in the coarse-graining histories description[20]. We will reproduce succinctly the argument in[Ga09] in order to facilitate the comparison with the previous results.

Consider an idealised double slit system as sketched in Fig. 2 (left), where a particle is fired from an emitter E and can pass through slits A or B on a wall W before ending on the final screen S either at the detector D (which is located for instance on a dark fringe) or elsewhere, $\overline{D}$. In [Ga09] the interpretation of

[20] After reading this it should be clear that an analysis following similar arguments could be performed for the n-slit experiment or more complicated systems like Kochen-Specker system [Ga09, Chap. 2].

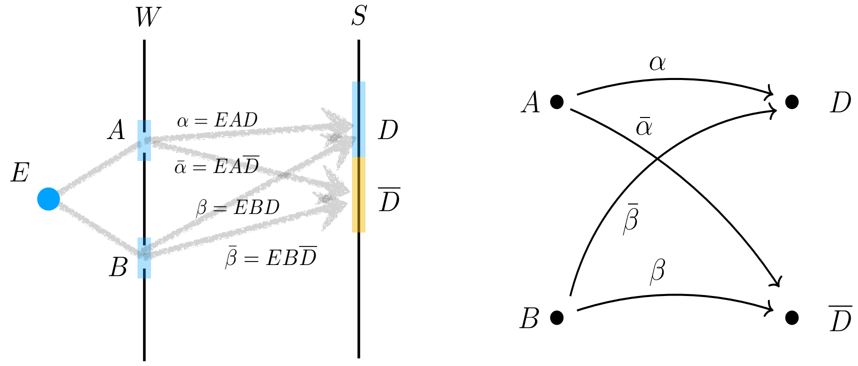

FIGURE 2. A coarse graining interpretation of the double slit experiment (left). The corresponding double slit quiver: U_2 (right).

the system is provided in terms of a Hilbert space, initial vector state $|\Psi\rangle$ and projectors P_A corresponding to finding the particle at slit A, P_B, P_D in a similar fashion an $P_{\overline{D}} = I - P_D$. We are not interested in such analysis here as we want to provide an algebraic description of it in terms of the structures discussed in the groupoid formalism. For that we will identify a family of events given by E, the 'emitter', A and B, determining if the particle pass through the slits A or B respectively, and D, $\overline{D}$, corresponding to the particle hitting the region D in the screen or $\overline{D}$. Hence the space of events, in this coarse-grained description of the system, is finite with 5 elements $\Omega = \{A, B, D, \overline{D}, E\}$. Note that in this picture the notion of event is not related to a complete family of compatible measurements. We do not even assume that there are actual detectors at the slits, but we are considering that it would be possible to determine that the particle is located near A precisely enough to discard that it would close to B and conversely[21][22].

The physical transitions of the system include the histories (see Fig. 2) $\alpha = EAD$, indicating by that the transition $\alpha\colon A \to D$, such that the fired particle cause the event A and consecutively D; $\beta = EBD$, that is, the transition $\beta\colon B \to D$ represents the histories that cause the event B and then D. Apart from α and β, there are two more transitions $\bar{\alpha} = EA\overline{D}\colon A \to \overline{D}$, $\bar{\beta} = EB\overline{D}\colon B \to \overline{B}$ with similar meaning (notice that $\bar{\alpha} \neq \alpha^{-1}$). The collection of transitions $U_2 = \{\alpha, \bar{\alpha}, \beta, \bar{\beta}\}$[23] do not define a groupoid but rather a quiver (see Fig. 2 right for the pictorical representation of it) and they correspond to the family of coarse-grained histories in [Ga09] description. From this point of view, notice that the event E is unnecessary

[21]This conception of events are closely related to the idealized notion of measurement used by Feynman to talk about trajectories of particles and to the notion of events in Sorkin's approach.

[22]Recall the notion of "beables".

[23]The notation U_2 corresponds to the notion of 'utility' graph with two companies used in graph theory.

since all relevant physical transitions assume that the particle has been fired, thus we assume that the space of events is $\Omega = \{A, B, D, \overline{D}\}$.

The quiver U_2 generates a groupoid $\mathbf{G}(U_2)$ by adding the units $1_A, 1_B, 1_D, 1_{\overline{D}}$, the inverses $\alpha^{-1}, \bar{\alpha}^{-1}, \beta^{-1}, \bar{\beta}^{-1}$ and four more transitions corresponding to $\gamma_{BA} = \beta^{-1} \circ \alpha \colon A \to B$, etc. Thus the order of the groupoid $\mathbf{G}(U_2)$ is 16 and it can be identified with the groupoid of pairs of Ω. Of course, we may argue on the physical meaning of the transitions γ_{BA}, $\gamma_{D\overline{D}}$,... as well as the inverses α^{-1}, β^{-1},.... There are no physical reasons to exclude them. Feynman's microscopic reversibility principle should imply the consideration of the transitions α^{-1}, ..., in the analysis of the system and then, because of logical consistency, of the transitions γ_{AB}, ... There is however no reason to consider states of the system where such transitions could actually happen, that is, they can be precluded so that the quantum measure describing the statistical properties of the system takes the value zero on them. This is exactly the point of view that we will take in our analysis, thus we will construct various states of the system possessing this property.

The construction of a quantum measure on $\mathbf{G}(U_2)$ considered as a coarse-grained histories description of the actual system is associated to a state on the algebra $\mathbb{C}[\mathbf{G}(U_2)]$ of the groupoid. In particular local states, which are the ones that lead to a dynamical interpretation of the theory, have associated characteristic functions of the form $\varphi \colon \mathbf{G}(U_2) \to \mathbb{C}$ of the form: $\varphi = e^{is}$ (up to a normalization factor), with s an action functional on the groupoid. In our case, because $\mathbf{G}(U_2)$ is generated by the utility quiver U_2, then it suffices to give the values of s on the transitions $\alpha, \bar{\alpha}, \beta, \bar{\beta}$, that is in the histories EAD, $EA\overline{D}$, EBD, and $EB\overline{D}$. Thus we may assume that:

$$s(\alpha) = s(\beta) + \delta = S_1 \,, \qquad s(\bar{\alpha}) = s(\bar{\beta}) = S_2 \,,$$

where δ, is a phase related to the difference between the physical paths that the particle follows when following the trajectories EAD and EBD respectively. A similar phase could be introduced in the action for $\bar{\alpha}$ and $\bar{\beta}$, however because of the particular configuration of the experiment, they have been chosen to be equal.

Notice that the values of s in all other transitions are determined by the properties of action functionals, for instance $s(1_A) = 0$, $s(\alpha^{-1}) = -S_1$, and so on. In particular $s(\gamma_{BA}) = s(\beta^{-1} \circ \alpha) = e^{i\delta}$. Thus the decoherence functional defined by the characteristic function φ is codified in a 16×16 matrix D whose entries are given by the numbers $D(\alpha_i, \alpha_j)$. If we concentrate ourselves in the 4×4 submatrix D_{U_2} corresponding to the quiver U_2, that is the transitions $\alpha, \bar{\alpha}, \beta, \bar{\beta}$, we will get:

$$D_{U_2} = \frac{1}{16} \begin{bmatrix} 1 & e^{i\delta} & 0 & 0 \\ e^{-i\delta} & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} .$$

Then, in the particular instance of $\delta = \pi$, we will have the matrix describing the quantum measure μ associated to the standard double slit experiment interpretation. Notice that in such case the measure of the set $\mathbf{V} = \{\alpha = EAD, \beta = EBD\}$ is given by:

$$\mu(\mathbf{V}) = D_{U_2}(\mathbf{V}, \mathbf{V}) = \sum_{\alpha_1, \alpha_2 \in \mathbf{V}} D_{U_2}(\alpha_1, \alpha_2) = 0\,,$$

in accordance with the fact the the detector is in a dark fringe, that is, the arrival of particles to it is precluded.

Notice that we may change the outcomes, either by moving the wall or the detectors, that is, we modify the outcomes $A', B', D', \overline{D}'$ and the transitions $\alpha', \beta', \bar{\alpha}', \bar{\beta}'$. Then the description of state will change accordingly. Notice that there will be an isomorphism τ from the group algebra of the original groupoid $\mathbf{G}(U_2)$ and the one obtained by using the primed data.

Notice that the transitions ending at D have no interference with histories ending at $\overline{D}$. This is a general feature of the groupoid formulation and is due to composability condition among them[24].

7. Conclusions and discussion

A unified description of Feynman's composition law for amplitudes and Schwinger's transformation functions is provided within the groupoid framework of Quantum Mechanics recently developed. An analysis of the statistical interpretation of the formalism is provided using as a fundamental notion the algebra of virtual transitions of the system (the groupoid algebra) and their states. Actually, it is shown that any state on the algebra of virtual transitions defines a decoherence functional (by means of the corresponding smeared character) and consequently a grade-2 measure, or a quantum measure in Sorkin's statistical intepretation of quantum mechanics. Then, either by starting from a quantum measure, or a state on the algebra of virtual transitions, there is a natural notion of amplitudes, called in the text *transition amplitudes*, which subsume the statistical interpretation of the theory. The groupoids based formalism provides a 'sum-over-histories'-like formula to compute the transition amplitudes and a natural theory of their representations in terms of vector-valued measures.

The states, or decoherence functionals, leading to Feynman's composition law are indentified as idempotent positive semidefinite function on the groupoid and, moreover, a natural multiplicative condition, isolates those states whose amplitudes satisfy Dirac-Feynamn's principle, that is, they have the form $\exp is$, with a s an action-like function defined on the groupoid of transitions. Such states, called *local* in the text, can be given a dynamical interpretation using a dynamical

[24]In the Hilbert space description of histories, this is due to the presence of a final time projector. The reader may wish to read [Ga09, Chap. 1.3] for a detailed description in the Hilbert space formalism of the coarse grained interpretation of the double slit experiment.

principle for the action function s as in Schwinger's original setting or, alternatively, by using Feynman's construction of the wave function and the corresponding Schrödinger's equation. These ideas, will be explored and will constitute the main argument of a forthcoming work.

Suggested by the work developed in this paper is a histories interpretation of the groupoid formalism. The notion of transition, the abstract Schwinger's notion of selective measurement that ch anges the state of the system, has a clear dynamical meaning, however, in Schwinger's conceptualisation, such transitions are elementary and no subjected to further scrutiny, while a dynamical description of the change of a system involves an analysis, that is, a decomposition of such change. This suggest a histories-based approach to the groupoid formalism, where the composition of transitions would be interpreted dynamically. In this sense the formalism described in the present paper can be understood as a coarse-grained histories interpretation of Schwinger's algebra where only the sources and targets, i.e., the events of the theory, are selected. Then a fine-grained histories description of the theory is needed to provide a proper interpretation of the dynamical nature of the aforementioned local states, and then, of Schwinger's dynamical principle. As commented before this will be the objective of another work.

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