Universidad de Burgos
Mathematical Physics Group

Lorentzian Poisson homogeneous spaces

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Outline

1. Motivation: Noncommutative spacetimes
2. Poisson-Lie groups and Poisson homogeneous spaces
3. (2+1) Poincaré PHS from Drinfel'd doubles
4. A Lorentzian PHS in (3+1) dimensions
5. Open problems

Hypothesis:

'Noncommutative' space-times can be used to describe relevant physical phenomena at the Planck scale (very high energy/curvature).

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- They give rise to **Planck scale uncertainty relations** when simultaneous measurements of 'spacetime observables' described by a noncommutative (C^*) algebra are considered:

$$[\hat{x}^i, \hat{x}^j]_{\lambda_P} \neq 0 \qquad \Longrightarrow \qquad \Delta\hat{x}^i \, . \, \Delta\hat{x}^j \geq \frac{1}{2} |\langle [\hat{x}^i, \hat{x}^j]_{\lambda_P} \rangle| > 0$$

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- This 'quantum' spacetime algebra should be a 'Planck scale' (λ_P) **deformation** of the 'classical' commutative spacetime:

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Many different physical contexts since Snyder (1947): nc field theory, string spacetimes, (2+1) quantum gravity, DSR and modified dispersion relations... Also different mathematical approaches to noncommutative geometry.
[see R. J. Szabo, Physics Reports 378, 207-299 (2003)].

There are many possibilities in order to introduce noncommutative algebras of spacetime 'coordinates'.

A mathematically consistent one is provided by **quantum groups**:
Deformations G_q of the Hopf algebra of functions on Lie groups or, equivalently, Hopf algebra deformations $U_q(\mathfrak{g})$ of the universal enveloping algebra of $\mathfrak{g} = \mathrm{Lie}(G)$ [Drinfel'd, Manin, Woronowicz].

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- The 'quantum' deformation parameter would depend on some Planck scale parameter: $q = q(\lambda_P)$ such that $\lim_{\lambda_P \to 0} q = 1$.
- In particular, **quantum (Anti)-de Sitter and Minkowski spacetimes** can be constructed as **quantum homogeneous spaces** of the corresponding quantum kinematical groups $SO_q(4,1)$, $SO_q(3,2)$ and $ISO_q(3,1)$.

Maximally symmetric spacetimes (AdS, dS, Minkowski) are **classical homogeneous spaces** $M = G/H$, where H is the Lorentz isotropy subgroup $SO(3,1)$ and G is, respectively, $SO(3,2)$, $SO(4,1)$ and $ISO(3,1)$.

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- Since H is a subgroup of G, the deformation H_q is fixed by the deformation G_q.
- Imposing that H_q **is a quantum subgroup** (*i.e.*, a Hopf subalgebra of G_q) turns out to be **too restrictive**.
  [Dijkhuizen and Koornwinder, 1994]

Why Poisson homogeneous spaces

Main results: [Drinfel'd]

Quantum groups G_q are **quantizations of Poisson-Lie groups** (G, Π) and **quantum homogeneous spaces** M_q are **quantizations of Poisson homogeneous spaces** (M, π).

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- Poisson-Lie groups (G, Π) are in one-to-one correspondence with **Lie bialgebra structures** $(\mathfrak{g}, \delta)$.
- Poisson homogeneous spaces (M, π) are in one-to-one correspondence with **with Lagrangian Lie subalgebras** ℓ **of the Drinfel'd double Lie algebra** $D(\mathfrak{g})$ associated to δ.

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- In order to be able to obtain the π bracket through canonical projection from the PL bracket Π, the isotropy subgroup H has to be **coisotropic with respect to δ**:

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- A particular case: when H is a **Poisson-Lie subgroup**

$$\delta\left(\mathfrak{h}\right) \subset \mathfrak{h} \wedge \mathfrak{h},$$

  its quantization would give rise to a quantum subgroup H_q.

As a consequence, for **the same 'clasical' $M = G/H$ we have different admissible Poisson (noncommutative) homogeneous structures**.

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1. We have **as many possible PL structures Π on G as Lie bialgebra structures (g, δ)**, whose dual δ^* is just the linearization of Π:

$$\{x, x\}_\Pi = A(x, \xi) \qquad \{x, \xi\}_\Pi = B(x, \xi) \qquad \{\xi, \xi\}_\Pi = C(x, \xi).$$

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The **coisotropy condition** ensures that π **can be obtained as the projection of Π onto the PHS coordinates**:

$$\{x, x\}_\pi = \{x, x\}_\Pi = A(x, 0) = F(x).$$

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Construct (A)dS and Minkowskian PHS and analyse their properties as a preliminary step for the construction of their associated QHS.

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  - Poincaré Lie algebra [Stachura, 1998]
  - dS and AdS Lie algebras [Zakrzewski 1994, Borowiec, Lukierski and Tolstoy, 2016]

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These classifications are **not written in a suitable kinematical basis** $\{J, K, P\}$ giving rise to the PHS with usual x^{μ} Minkowski coordinates.

In particular:

Some examples of Lorentzian PHS:

- In (2+1) dimensions, Minkowski PHS coming from all possible Drinfel'd double structures of the (2+1) Poincaré Lie algebra are constructed. The DD (A)dS Poisson-Lie groups are known. [1]
- A PHS for the (3+1)-dimensional AdS group with respect to the so-called κ-Poisson–Lie structure is explicitly given.

  Its Minkowskian limit can be obtained when the cosmological constant vanishes.

[1] A.B., F.J. Herranz, C. Meusburger, Class. Quantum Grav. 30 (2013), 155012

2. Poisson-Lie groups and Poisson homogeneous spaces

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Theorem [Drinfel'd]. Let G be a Lie group with Lie algebra $\mathfrak{g}$:

a) If (G, Π) is a Poisson-Lie group, then $\mathfrak{g}$ has a natural Lie bialgebra structure $(\mathfrak{g}, \delta)$, called the tangent Lie bialgebra of G.
b) Conversely, if G is connected and simply connected, every Lie bialgebra structure $(\mathfrak{g}, \delta)$ is the tangent Lie bialgebra of a unique Poisson structure on G which makes G into a Poisson-Lie group.

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The **cocommutator map** $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ is such that:

- i) δ is a **1–cocycle**, i.e.,
  $\delta([X, Y]) = [\delta(X), 1 \otimes Y + Y \otimes 1] + [1 \otimes X + X \otimes 1, \delta(Y)], \; \forall X, Y \in g.$
- ii) The **dual map** $\delta^* : \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*$ is a **Lie bracket** on $\mathfrak{g}^*$.

- In some cases the 1-cocycle δ is a **coboundary**

$$\delta(X) = [1 \otimes X + X \otimes 1,\ r] \qquad X \in \mathfrak{g}$$

where r is a skewsymmetric element of $\mathfrak{g} \otimes \mathfrak{g}$ (the ***r*-matrix**)

$$r = r^{ab}\, X_a \wedge X_b$$

which has to be a solution of the **modified classical Yang–Baxter equation** (mCYBE)

$$[X \otimes 1 \otimes 1 + 1 \otimes X \otimes 1 + 1 \otimes 1 \otimes X, [[r, r]]\,] = 0 \qquad X \in \mathfrak{g}.$$

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- For a coboundary Poisson-Lie group G, the Poisson structure Π on G is given by the so-called **Sklyanin bracket**

$$\{f, g\} = r^{ij}\left(\nabla_i^L f\, \nabla_j^L g - \nabla_i^R f\, \nabla_j^R g\right), \qquad f, g \in \mathcal{C}^\infty(G)$$

where ∇_i^L, ∇_i^R are left- and right-invariant vector fields on G.

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The **Drinfel'd double Lie algebra** $D(\mathfrak{g})$ associated to the Lie bialgebra $(\mathfrak{g}, \delta)$ is the Lie algebra structure on the vector space $\mathfrak{g} + \mathfrak{g}^*$ given by

$$[X_i, X_j] = C_{ij}^k X_k, \qquad [x^i, x^j] = f_k^{ij} x^k, \qquad [x^i, X_j] = C_{jk}^i x^k - f_j^{ik} X_k,$$

where $\{X_1, \dots, X_n\}$ and $\{x^1, \dots, x^n\}$ are dual basis for $\mathfrak{g}$ and $\mathfrak{g}^*$, under the canonical pairing

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$D(\mathfrak{g})$ has a **canonical quasi-triangular Lie bialgebra structure** given by

$$r = \sum_i x^i \otimes X_i,$$

since r is always a solution of the CYBE $[[r, r]] = 0$.

Let (G, Π) **be a PL group**. A **Poisson homogeneous space** is a Poisson manifold (M, π) with **a transitive group action** $\triangleright : G \times M \to M$ **that is a Poisson map** with respect to the Poisson structure π on M and the product $\Pi \times \pi$ of the Poisson structures on G and M.

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Theorem [Drinfel'd]

Let G be a Poisson-Lie group and let H be a connected subgroup of G. Poisson homogeneous spaces on $M = G/H$ are in **one-to one correspondence with Lagrangian Lie subalgebras** ℓ **of the double Lie algebra** $D(\mathfrak{g})$ such that $\ell \cap \mathfrak{g} = \mathfrak{h}$.

The Lagrangian Lie subalgebra ℓ of the double Lie algebra $D(\mathfrak{g})$ fulfills:

- Maximality: $\dim \ell = \dim \mathfrak{g} \equiv N$.
- Isotropy: $\langle \ell, \ell \rangle = 0$.
- The condition $\ell \cap \mathfrak{g} = \mathfrak{h}$.

3. Poincaré PHS from Drinfel'd doubles in (2+1)

The **(2+1) Poincaré Lie algebra**:

$$
\begin{aligned}
&[J, K_1] = K_2, && [J, K_2] = -K_1, && [K_1, K_2] = -J, \\
&[J, P_0] = 0, && [J, P_1] = P_2, && [J, P_2] = -P_1, \\
&[K_1, P_0] = P_1, && [K_1, P_1] = P_0, && [K_1, P_2] = 0, \\
&[K_2, P_0] = P_2, && [K_2, P_1] = 0, && [K_2, P_2] = P_0, \\
&[P_0, P_1] = 0, && [P_0, P_2] = 0, && [P_1, P_2] = 0,
\end{aligned}
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Two quadratic Casimirs:

$$C_1 = P_0^2 - P_1^2 - P_2^2,$$

$$C_2 = J\,P_0 + K_2\,P_1 - K_1\,P_2.$$

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(2+1) gravity can be described as a Chern-Simons gauge theory whose symplectic structure can be defined in terms of PL structures [Fock and Rosly 1992, Alekseev and Malkin 1995], and the most natural PL structures are those coming from classical Drinfel'd doubles [Meusburger and Schroers, 2009].

$$[Y_i, Y_j] = C^k_{ij} Y_k \qquad [y^i, y^j] = f^{ij}_k y^k \qquad [y^i, Y_j] = C^i_{jk} y^k - f^{ik}_j Y_k$$

Table 1: The eigth non-equivalent DD Lie algebras which are isomorphic to the (2+1) Poincaré algebra. The parameter ω can be rescaled to any non-zero real number of the same sign, while λ is an essential parameter. In Case 5 they must obey $\omega\lambda > 0$. In Case 6 we have $\omega > 0$.

	Case 0	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6	Case 7
$[Y_0, Y_1]$	$2Y_1$	Y_1	Y_1	Y_1	$-Y_2$	Y_1	Y_1	$-Y_2$
$[Y_0, Y_2]$	$-2Y_2$	Y_2	Y_2	$Y_1 + Y_2$	Y_1	$Y_1 + Y_2$	Y_2	Y_1
$[Y_1, Y_2]$	Y_0	0	0	0	0	0	0	0
$[y^0, y^1]$	0	y^0	0	λy^2	0	λy^2	0	0
$[y^0, y^2]$	0	y^1	y^1	0	$-y^0$	0	y^1	$-y^0$
$[y^1, y^2]$	0	y^2	0	0	$\lambda y^0 - y^1$	$2\omega y^0$	$2\omega y^0$	$-y^1$
$[y^0, Y_0]$	0	$-Y_1$	0	0	$-Y_2$	0	0	Y_2
$[y^0, Y_1]$	y^2	$-Y_2$	$-Y_2$	0	$\lambda Y_2 - y^2$	0	$-Y_2$	0
$[y^0, Y_2]$	$-y^1$	0	0	$-\lambda y^1$	$Y_0 - \lambda Y_1 + y^1$	$-\lambda Y_1$	0	0
$[y^1, Y_0]$	$2y^1$	$Y_0 + y^1$	y^1	$y^1 + y^2$	0	$-2\omega Y_2 + y^1 + y^2$	$-2\omega Y_2 + y^1$	y^2
$[y^1, Y_1]$	$-2y^0$	$-y^0$	$-y^0$	$-y^0$	$-Y_2$	$-y^0$	$-y^0$	Y_2
$[y^1, Y_2]$	0	$-Y_2$	0	$\lambda Y_0 - y^0$	$Y_1 - y^0$	$\lambda Y_0 - y^0$	0	$-y^0$
$[y^2, Y_0]$	$-2y^2$	y^2	y^2	y^2	0	$2\omega Y_1 + y^2$	$2\omega Y_1 + y^2$	$-Y_0 - y^1$
$[y^2, Y_1]$	0	Y_0	Y_0	0	y^0	0	Y_0	$-Y_1 + y^0$
$[y^2, Y_2]$	$2y^0$	$Y_1 - y^0$	$-y^0$	$-y^0$	0	$-y^0$	$-y^0$	0

A.B., I. Gutiérrez-Sagredo, F.J. Herranz, *The Poincaré group as a Drinfel'd double*, preprint.

- A **generic element of the $P(2+1)$ group** is constructed as

$$G = \exp\left(x^0\,\rho(P_0)\right)\exp\left(x^1\,\rho(P_1)\right)\exp\left(x^2\,\rho(P_2)\right)\exp\left(\xi^1\,\rho(K_1)\right)\exp\left(\xi^2\,\rho(K_2)\right)\exp\left(\theta\,\rho(J)\right).$$

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$$G = \exp\left(x^0\,\rho(P_0)\right)\exp\left(x^1\,\rho(P_1)\right)\exp\left(x^2\,\rho(P_2)\right)\exp\left(\xi^1\,\rho(K_1)\right)\exp\left(\xi^2\,\rho(K_2)\right)\exp\left(\theta\,\rho(J)\right).$$

- $\{x^0, x^1, x^2\}$ are **local coordinates on the homogeneous Minkowski spacetime** defined as $M = G/H$ with $H = \{J, K_1, K_2\}$.

- A **generic element of the $P(2+1)$ group** is constructed as

$$G = \exp\left(x^0\,\rho(P_0)\right)\exp\left(x^1\,\rho(P_1)\right)\exp\left(x^2\,\rho(P_2)\right)\exp\left(\xi^1\,\rho(K_1)\right)\exp\left(\xi^2\,\rho(K_2)\right)\exp\left(\theta\,\rho(J)\right).$$

- $\{x^0, x^1, x^2\}$ are **local coordinates on the homogeneous Minkowski spacetime** defined as $M = G/H$ with $H = \{J, K_1, K_2\}$.
- Left and right invariant vector fields on $P(2+1)$ are:

$$\begin{aligned}
\nabla^L_J &= \partial_\theta,\\
\nabla^L_{K_1} &= \frac{\cos\theta}{\cosh\xi^2}\left(\partial_{\xi^1} + \sinh\xi^2\,\partial_\theta\right) + \sin\theta\,\partial_{\xi^2},\\
\nabla^L_{K_2} &= -\frac{\sin\theta}{\cosh\xi^2}\left(\partial_{\xi^1} + \sinh\xi^2\,\partial_\theta\right) + \cos\theta\,\partial_{\xi^2},\\
\nabla^L_{P_0} &= \cosh\xi^2\left(\cosh\xi^1\partial_{x^0} + \sinh\xi^1\partial_{x^1}\right) + \sinh\xi^2\partial_{x^2},\\
\nabla^L_{P_1} &= \cos\theta\left(\sinh\xi^1\partial_{x^0} + \cosh\xi^1\partial_{x^1}\right) + \sin\theta\left(\sinh\xi^2\left(\cosh\xi^1\partial_{x^0} + \sinh\xi^1\partial_{x^1}\right) + \cosh\xi^2\partial_{x^2}\right),\\
\nabla^R_{P_2} &= -\sin\theta\left(\sinh\xi^1\partial_{x^0} + \cosh\xi^1\partial_{x^1}\right) + \cos\theta\left(\sinh\xi^2\left(\cosh\xi^1\partial_{x^0} + \sinh\xi^1\partial_{x^1}\right) + \cosh\xi^2\partial_{x^2}\right),
\end{aligned}$$

$$\begin{aligned}
\nabla^R_J &= -x^2\partial_{x^1} + x^1\partial_{x^2} + \frac{\cosh\xi^1}{\cosh\xi^2}\left(\partial_\theta - \sinh\xi^2\partial_{\xi^1}\right) + \sinh\xi^1\partial_{\xi^2},\\
\nabla^R_{K_1} &= x^1\,\partial_{x^0} + x^0\,\partial_{x^1} + \partial_{\xi^1},\\
\nabla^R_{K_2} &= x^2\partial_{x^0} + x^0\partial_{x^2} + \frac{\sinh\xi^1}{\cosh\xi^2}\left(-\sinh\xi^2\partial_{\xi^1} + \partial_\theta\right) + \cosh\xi^1\partial_{\xi^2},\\
\nabla^R_{P_0} &= \partial_{x^0}, \qquad \nabla^R_{P_1} = \partial_{x^1}, \qquad \nabla^R_{P_2} = \partial_{x^2}.
\end{aligned}$$

Table 2: The (2+1) Poincaré r-matrices and Poisson subgroup/coisotropy condition for each of the eight DD structures on $\mathfrak{p}(2+1)$ as well as the corresponding class in the Stachura classification.

Case	Classical r-matrix r'_i	$\delta_D\,(\mathfrak{h})$	Stachura class.
0	$\frac{1}{2}(-P_0 \wedge J - P_1 \wedge K_2 + P_2 \wedge K_1)$	$= 0$	(IV)
1	$K_1 \wedge J + K_1 \wedge K_2 + (-P_0 \wedge J - P_1 \wedge K_2 + P_2 \wedge K_1)$	$\subset \mathfrak{h} \wedge \mathfrak{h}$	(I)
2	$P_2 \wedge J - P_0 \wedge K_2 - P_2 \wedge K_2 + \frac{1}{2}(P_0 \wedge J - P_1 \wedge K_2 + P_2 \wedge K_1)$	$\subset \mathfrak{h} \wedge \mathfrak{g}$	(IIa)
3	$-P_2 \wedge J - P_0 \wedge K_2 - P_2 \wedge K_2 + \frac{1}{2}(-P_0 \wedge J + P_1 \wedge K_2 - P_2 \wedge K_1)$ $+\frac{1}{\lambda}(P_0 \wedge P_1 + 2(P_0 \wedge P_2 + P_2 \wedge P_1))$	$\not\subset \mathfrak{h} \wedge \mathfrak{g}$	(IIa)
4	$P_2 \wedge J + \frac{1}{2}(P_0 \wedge J - P_1 \wedge K_2 + P_2 \wedge K_1) + \lambda P_0 \wedge P_2$	$\not\subset \mathfrak{h} \wedge \mathfrak{g}$	(IIIb)
5	$P_1 \wedge J + \frac{1}{2}(-P_0 \wedge J + P_1 \wedge K_2 - P_2 \wedge K_1) + \frac{1}{\lambda} P_1 \wedge P_0$	$\not\subset \mathfrak{h} \wedge \mathfrak{g}$	(IIIb)
6	$P_0 \wedge K_2 + \frac{1}{2}(-P_0 \wedge J + P_1 \wedge K_2 + P_2 \wedge K_1)$	$\subset \mathfrak{h} \wedge \mathfrak{g}$	(IIIb)
7	$P_2 \wedge J + \frac{1}{2}(-P_0 \wedge J + P_1 \wedge K_2 - P_2 \wedge K_1)$	$\subset \mathfrak{h} \wedge \mathfrak{g}$	(IIIb)

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Among the eight ***r*-matrices generating the (2+1) PL structures**:

- We have **five coisotropic cases** with respect to the Lorentz isotropy subalgebra.
- Cases 0 and 1 are of **Poisson subgroup** type.
- All the r-matrices should be multiplied by a constant (λ_P).

DD Poisson Minkowski spacetimes in (2+1) dimensions

Case 0

$$\{x^0, x^1\} = -x^2, \qquad \{x^0, x^2\} = x^1, \qquad \{x^1, x^2\} = x^0,$$

Case 1

$$\begin{aligned}\{x^0, x^1\} &= -x^2(x^0 + x^1) + 2x^2,\\ \{x^0, x^2\} &= x^1(x^0 + x^1) - 2x^1,\\ \{x^1, x^2\} &= x^0(x^0 + x^1) - 2x^0,\end{aligned}$$

Case 2

$$\{x^0, x^1\} = 0, \qquad \{x^0, x^2\} = -\left(x^0 - x^2\right), \qquad \{x^1, x^2\} = -\left(x^0 - x^2\right),$$

Case 6

$$\{x^0, x^1\} = 0, \qquad \{x^0, x^2\} = -x^0 + x^1, \qquad \{x^1, x^2\} = 0,$$

Case 7

$$\{x^0, x^1\} = 0, \qquad \{x^0, x^2\} = 0, \qquad \{x^1, x^2\} = -(x^0 + x^2).$$

Quantization is **non-trivial for Case 2** (quadratic Poisson algebra).

4. A Lorentzian PHS in (3+1) dimensions

The **(3+1)D AdS$_\omega$ Lie algebra** ($\omega = -\Lambda$):

$$
\begin{array}{lll}
[J_a, J_b] = \epsilon_{abc} J_c \,, & [J_a, P_b] = \epsilon_{abc} P_c \,, & [J_a, K_b] = \epsilon_{abc} K_c \,, \\
[K_a, P_0] = P_a \,, & [K_a, P_b] = \delta_{ab} P_0 \,, & [K_a, K_b] = -\epsilon_{abc} J_c \,, \\
[P_0, P_a] = \omega K_a \,, & [P_a, P_b] = -\omega \epsilon_{abc} J_c \,, & [P_0, J_a] = 0 \,.
\end{array}
$$

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&[K_a, P_0] = P_a\,, && [K_a, P_b] = \delta_{ab} P_0\,, && [K_a, K_b] = -\epsilon_{abc} J_c\,,\\
&[P_0, P_a] = \omega K_a\,, && [P_a, P_b] = -\omega \epsilon_{abc} J_c\,, && [P_0, J_a] = 0\,.
\end{aligned}
$$

Explicilty, $\mathbf{AdS}_\omega^{3+1}$ comprises the three following Lorentzian spacetimes:

- $\omega > 0, \Lambda < 0$: AdS spacetime $\mathbf{AdS}^{3+1} \equiv \mathrm{SO}(3,2)/\mathrm{SO}(3,1)$.
- $\omega < 0, \Lambda > 0$: dS spacetime $\mathbf{dS}^{3+1} \equiv \mathrm{SO}(4,1)/\mathrm{SO}(3,1)$.
- $\omega = \Lambda = 0$: Minkowski spacetime $\mathbf{M}^{3+1} \equiv \mathrm{ISO}(3,1)/\mathrm{SO}(3,1)$.

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- $\omega = \Lambda = 0$: Minkowski spacetime $\mathbf{M}^{3+1} \equiv \mathrm{ISO}(3,1)/\mathrm{SO}(3,1)$.

Casimir operators: the quadratic one

$$
\mathcal{C} = P_0^2 - \mathbf{P}^2 + \omega \left(\mathbf{J}^2 - \mathbf{K}^2 \right)
$$

and the quartic one (Pauli-Lubanski)

$$
\mathcal{W} = W_0^2 - \mathbf{W}^2 + \omega \, (\mathbf{J} \cdot \mathbf{K})^2
$$

$$
W_0 = \mathbf{J} \cdot \mathbf{P} \qquad W_a = -J_a P_0 + \epsilon_{abc} K_b P_c
$$

The **classical r-matrix for the AdS κ-deformation** is [2]

$$r = z\left(K_1 \wedge P_1 + K_2 \wedge P_2 + K_3 \wedge P_3 + \sqrt{\omega}\, J_1 \wedge J_2\right), \qquad z = 1/\kappa,$$

[2] A.B., F.J. Herranz, F. Musso, P. Naranjo, PLB (2017); arXiv:1612.03169

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and the **cocommutator map** reads

$$\begin{aligned}
&\delta(P_0) = 0, \qquad \delta(J_3) = 0,\\
&\delta(J_1) = z\sqrt{\omega}\, J_1 \wedge J_3, \qquad \delta(J_2) = z\sqrt{\omega}\, J_2 \wedge J_3,\\
&\delta(P_1) = z\left(P_1 \wedge P_0 - \omega\, J_2 \wedge K_3 + \omega\, J_3 \wedge K_2 + \sqrt{\omega}\, J_1 \wedge P_3\right),\\
&\delta(P_2) = z\left(P_2 \wedge P_0 - \omega\, J_3 \wedge K_1 + \omega\, J_1 \wedge K_3 + \sqrt{\omega}\, J_2 \wedge P_3\right),\\
&\delta(P_3) = z\left(P_3 \wedge P_0 - \omega\, J_1 \wedge K_2 + \omega\, J_2 \wedge K_1 - \sqrt{\omega}\, J_1 \wedge P_1 - \sqrt{\omega}\, J_2 \wedge P_2\right),\\
&\delta(K_1) = z\left(K_1 \wedge P_0 + J_2 \wedge P_3 - J_3 \wedge P_2 + \sqrt{\omega}\, J_1 \wedge K_3\right),\\
&\delta(K_2) = z\left(K_2 \wedge P_0 + J_3 \wedge P_1 - J_1 \wedge P_3 + \sqrt{\omega}\, J_2 \wedge K_3\right),\\
&\delta(K_3) = z\left(K_3 \wedge P_0 + J_1 \wedge P_2 - J_2 \wedge P_1 - \sqrt{\omega}\, J_1 \wedge K_1 - \sqrt{\omega}\, J_2 \wedge K_2\right).
\end{aligned}$$

The **coisotropy condition** holds **$\delta(\mathfrak{h}) \subset \mathfrak{h} \wedge \mathfrak{g}$.**

[2] A.B., F.J. Herranz, F. Musso, P. Naranjo, PLB (2017); arXiv:1612.03169

By **computing the Sklyanin bracket** for the r-matrix and by taking the canonical projection onto the spacetime Poisson subalgebra we get the **(3+1) κ-AdS noncommutative spacetime** [3]

$$\{x_0, x_1\} = -\frac{z\tanh(\sqrt{\omega}\,x_1)\mathrm{sech}^2(\sqrt{\omega}\,x_2)\mathrm{sech}^2(\sqrt{\omega}\,x_3)}{\sqrt{\omega}} \qquad \omega \to 0 \qquad -z\,x_1$$

$$\{x_0, x_2\} = -\frac{z\tanh(\sqrt{\omega}\,x_2)\mathrm{sech}^2(\sqrt{\omega}\,x_3)}{\sqrt{\omega}} \qquad \omega \to 0 \qquad -z\,x_2$$

$$\{x_0, x_3\} = -\frac{z\tanh(\sqrt{\omega}\,x_3)}{\sqrt{\omega}} \qquad \omega \to 0 \qquad -z\,x_3$$

$$\{x_1, x_2\} = -\frac{z\cosh(\sqrt{\omega}\,x_1)\tanh^2(\sqrt{\omega}\,x_3)}{\sqrt{\omega}} \qquad \omega \to 0 \qquad 0$$

$$\{x_1, x_3\} = \frac{z\cosh(\sqrt{\omega}\,x_1)\tanh(\sqrt{\omega}\,x_2)\tanh(\sqrt{\omega}\,x_3)}{\sqrt{\omega}} \qquad \omega \to 0 \qquad 0$$

$$\{x_2, x_3\} = -\frac{z\sinh(\sqrt{\omega}\,x_1)\tanh(\sqrt{\omega}\,x_3)}{\sqrt{\omega}} \qquad \omega \to 0 \qquad 0$$

With respect to the Minkowski limit $\omega = -\Lambda \to 0$, space coordinates do not commute and space isotropy is lost.

[3] A.B., I. Gutiérrez-Sagredo, F.J. Herranz, *The κ-AdS Poisson homogeneous spacetime*, preprint.

The (2+1) κ-AdS$_\omega$ noncommutative spacetime

The **classical r-matrix** in the (2+1) case reads [4]

$$r = z\,(K_1 \wedge P_1 + K_2 \wedge P_2),$$

and the **Lie bialgebra** $(\mathrm{AdS}_\omega, \delta)$ is

$$\begin{aligned}
\delta(P_0) &= \delta(J) = 0,\\
\delta(P_1) &= z(P_1 \wedge P_0 - \omega\, K_2 \wedge J),\\
\delta(P_2) &= z(P_2 \wedge P_0 + \omega\, K_1 \wedge J),\\
\delta(K_1) &= z(K_1 \wedge P_0 + P_2 \wedge J),\\
\delta(K_2) &= z(K_2 \wedge P_0 - P_1 \wedge J).
\end{aligned}$$

This Lie bialgebra **fulfills the coisotropy condition** with $\mathfrak{h} = \{J, K_1, K_2\}$:

$$\delta(\mathfrak{h}) \subset \mathfrak{h} \wedge \mathfrak{g}.$$

[4] A.B., F.J. Herranz, M.A. del Olmo, M. Santander, J. Phys. A **27** (1994) 1283.

Poisson–Lie group relations among spacetime x_μ group coordinates:

$$\{x_0, x_1\} = -z\,\frac{\tanh\sqrt{\omega}x_1}{\sqrt{\omega}\cosh^2\sqrt{\omega}x_2} \qquad \omega \to 0 \qquad -z\,x_1$$

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The quantum AdS$_\omega$ group in 'local coordinates' would be the quantization of the above nonlinear PL bracket. In particular, since $\{x_1, x_2\} = 0$ we could write:

$$[\hat{x}_0, \hat{x}_1] = -z\,\frac{\tanh\sqrt{\omega}\hat{x}_1}{\sqrt{\omega}\cosh^2\sqrt{\omega}\hat{x}_2} = -z\left(\hat{x}_1 - \frac{1}{3}\omega\hat{x}_1^3 - \omega\hat{x}_1\hat{x}_2^2\right) + \mathcal{O}(\omega^2),$$

$$[\hat{x}_0, \hat{x}_2] = -z\,\frac{\tanh\sqrt{\omega}\hat{x}_2}{\sqrt{\omega}} = -z\left(\hat{x}_2 - \frac{1}{3}\omega\hat{x}_2^3\right) + \mathcal{O}(\omega^2),$$

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Open problems

- Construction of the **coisotropic PHS for AdS^{2+1}**.

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- The **Minkowski and (A)dS^{3+1} PHS:** the H subgroup is $SO(3,1)$. The complete classification of coisotropic and Poisson subgroup Lie bialgebra structures on $SO(3,2)$, $SO(4,1)$ and $P(3+1)$.

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- Construction of **PHS of time-like worldlines**:

$$\mathfrak{h} = \{J_1, J_2, J_3, P_0\}$$

$\omega = -\Lambda$	Space of time-like worldlines $\mathbb{S}_{(2)} \equiv \{x_1, x_2, x_3, \xi_1, \xi_2, \xi_3\}$
$\omega > 0,\ \Lambda < 0$	$\mathbf{LAdS}^{3\times 3} = SO(3,2)/\left(SO(3) \otimes SO(2)\right)$
$\omega = \Lambda = 0$	$\mathbf{LM}^{3\times 3} = ISO(3,1)/\left(SO(3) \otimes \mathbb{R}\right)$
$\omega < 0,\ \Lambda > 0$	$\mathbf{LdS}^{2\times 2} = SO(4,1)/\left(SO(3) \otimes SO(2,1)\right)$

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- **Quantization of PHS** and **representation theory** of the associated spacetime algebras.

THANKS FOR YOUR ATTENTION

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Quantized Space-Time

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(Received May 13, 1946)

It is usually assumed that space-time is a continuum. This assumption is not required by Lorentz invariance. In this paper we give an example of a Lorentz invariant discrete space-time.

for transformations from one inertial frame to another. It is usually assumed that the variables x, y, z, and t take on a continuum of values and that they may take on these values simultaneously. This last assumption we change to the following:

x, y, z, and t are Hermitian operators for the space-time coordinates of a particular Lorentz frame; the spectrum of each of the operators x, y, z, and t is composed of the possible results of a measurement of the corresponding quantity; the operators x, y, z, and t shall be such that the spectra of the operators x', y', z', t' formed by taking linear combinations of x, y, z, and t, which leave the quadratic form (1) invariant, shall be the same as the spectra of x, y, z, and t.

The ten operators defined in (3) and (4) have a total of forty-five commutators. Only six of these commutators differ from the ordinary ones and these six are

$$[x, y]=(ia^2/\hbar)L_z, \quad [t, x]=(ia^2/\hbar c)M_x,$$
$$[y, z]=(ia^2/\hbar)L_x, \quad [t, y]=(ia^2/\hbar c)M_y, \quad (5)$$
$$[z, x]=(ia^2/\hbar)L_y, \quad [t, z]=(ia^2/\hbar c)M_z.$$

We see from these commutators that if we take the limit $a\rightarrow 0$ keeping $\hbar$ and c fixed, our quantized space-time changes to the ordinary continuous space-time.