Gravitational Waves: some Geometric Properties

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Example

$(\mathcal{M}, g)$ semi–Riemannian manifold:

"straight line" $\approx$ geodesic

semi–Riemannian manifolds

Definition

A semi–Riemannian manifold is a smooth manifold $\mathcal{M}$, of dimension $n \geq 1$, endowed with a non–degenerate metric

$$g : \mathcal{M} \to T^*\mathcal{M} \otimes T^*\mathcal{M}$$

of constant index s_0, i.e., for each $x \in \mathcal{M}$

$$g(x)[\cdot, \cdot] : T_x\mathcal{M} \times T_x\mathcal{M} \to \mathbb{R}$$

is a scalar product of index s_0 on the tangent space $T_x\mathcal{M}$.

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  A spacetime is a connected time-oriented 4-dimensional Lorentzian manifold.

Geodesics

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Definition

A smooth curve $z : I \to \mathcal{M}$, I real interval, is a geodesic if its vector field is parallel or, equivalently, its acceleration vanishes:

$$D_s \dot{z}(s) = 0 \quad \text{for all } s \in I.$$

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Remark

If $z : I \to \mathcal{M}$ is a geodesic, then a constant $E_z \in \mathbb{R}$ exists such that

$$g(\dot{z}(s), \dot{z}(s)) \equiv E_z.$$

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When is $\mathcal{M}$ geodesically complete?
(i.e., every inextendible geodesic is defined on the entire real line)

Hopf–Rinow Theorem

Let $(\mathcal{M}_0, \langle\cdot,\cdot\rangle_R)$ be a connected smooth Riemannian manifold which is a metric space equipped with the distance associated to $\langle\cdot,\cdot\rangle_R$:

$$d(x_1, x_2) = \inf \left\{ \int_a^b \sqrt{\langle\gamma', \gamma'\rangle_R} ds : \ \gamma \in A_{x_1,x_2} \right\}$$

with x_1, $x_2 \in \mathcal{M}_0$ and $\gamma \in A_{x_1,x_2}$ if $\gamma : [a, b] \to \mathcal{M}_0$ is a piecewise smooth curve joining x_1 to x_2.

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$\mathcal{M}_0$ complete $\iff$ $\mathcal{M}_0$ geodesically complete.

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Indefinite manifolds

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Example (anti–de Sitter spacetime)

$$\mathcal{M} = \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\times \mathbb{R}$$

equipped with the Lorentzian metric

$$\langle \cdot, \cdot \rangle_L = \frac{1}{\cos^2 x} \left(dx^2 - dt^2\right).$$

$\mathcal{M}$ is geodesically complete but not geodesically connected.

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  Penrose singularity theorem: some sort of geodesic incompleteness occurs inside any black hole whenever matter satisfies reasonable energy conditions.

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Taking some special semi–Riemannian manifolds, e.g.

- warped product spacetimes,
- stationary spacetimes,
- orthogonal splitting spacetimes,
- Gödel type spacetimes,
- generalized plane waves,
- $\ldots$,

ad hoc techniques are developed.

Gravitational Waves

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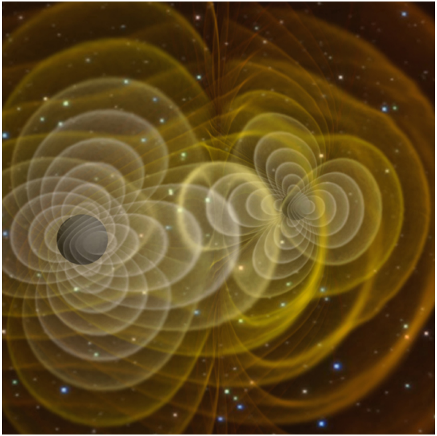

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"General relativity is a field theory and, roughly speaking, it does for the problem of gravitation what Maxwell's theory did for the problem of electromagnetic phenomena.
For this reason, gravitational waves can be deduced from general relativity just as the existence of electromagnetic waves can be deduced from Maxwell's theory."

Leonard Infeld, *Quest: An Autobiography* (1941)

Gravitational Waves

Einstein's mathematics showed that massive accelerating objects would disrupt space-time in such a way that 'waves' of distorted space would radiate from the source (like the movement of waves away from a stone thrown into a pond),

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i.e., gravitational waves are 'ripples' in the fabric of space-time caused by some of the most violent and energetic processes in the Universe, which travel at the speed of light and carry with them information about their cataclysmic origins, as well as invaluable clues to the nature of gravity itself.

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The strongest gravitational waves are produced by catastrophic events such as

- colliding black holes,
- the collapse of stellar cores (supernovae),
- coalescing neutron stars or white dwarf stars,
- ...

Do Gravitational Waves exist?

Astron. Nachr. / AN **326** (2005), No. 7 – Short Contributions AG 2005 Köln

Einstein and the Gravitational Waves

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In 1918 Einstein published the paper ÜBER GRAVITATIONSWELLEN [1] in which, for the first time, the effect of gravitational waves was calculated, resulting in his famous "quadrupole formula" (QF). Einstein was forced to this publication due to a serious error in his 1916 paper [2], where he had developed the linear approximation ("weak-field") scheme to solve the field equations of general relativity (GR). In analogy to electrodynamics, where accelerated charges emit electromagnetic waves, the linearized theory creates gravitational waves, popagating with the speed of light in the (background) Minkowski space-time. A major difference: Instead of a dipole moment, now a quadrupole moment is needed. Thus sources of gravitational waves are objects like a "rotating dumbbell", e. g. realized by a binary star system.
As there was no chance for detecting gravitational waves, due to their extreme weakness of the order $(\frac{v}{c})^5$, the theory advanced slow in the first decades. The existence of gravitational waves was always a matter of controversy. Curiously Einstein himself was not convinced in 1936. In a paper with Nathan Rosen he came to the conclusion, that gravitational waves do not exist! Curiously too is the story of its publication. Einstein's manuscript, titled DO GRAVITATIONAL WAVES EXIST?, was rejected by the "Physical Review". In an angry reply he withdrawed the paper, to appear later in the "Journal of the Franklin Institute" (choosing a less provoking headline [3]).

Do Gravitational Waves exist?

To clear the situation, various approximation schemes were developed. One of the first, introduced by Einstein, Infeld and Hoffmann in 1938 [4], led to the famous EIH equations. This "post-Newtonian" treatment describes slow moving bodies in a weak field ("bounded systems"). In the EIH approximation there is no radiation up to the order $(\frac{v}{c})^4$, the energy remains constant. The QF appears in the next order, as demonstrated by Hu in 1947 [5]. What's about fast moving particles? This problem had to wait until the early 1960's, when the Lorentz-invariant perturbation methods ("fast-motion approximation"), describing "unbounded systems", were developed. The question of an analogy to the QF ("radiation damping") was strongly discussed.
In 1975 a major boost was caused by the discovery of the binary pulsar PSR 1913 + 16 by Hulse and Taylor [6]. Over the next years their data showed a decrease of the period of revolution – as predicted by the QF! But this (indirect) proof – in the "bounded" case – did not stop the controversy: On the contrary, the fight gets even stronger. The different approximation formalisms where criticized by Ehlers, Havas and others [7]. The basic difficulties are: (1) In contrast to electrodynamics, the equations of motion in GR are not a separate part of the theory, but already inherent in the field equations. (2) GR is an essential non-linear theory. Any approximation must treat these facts carefully. After a phase of clarification, introducing new methods (e. g. asymptotic field conditions, post-linear approximations), the believe in gravitational waves, and especially in Einstein's QF, is now stronger than ever – eventually visible in expensive terrestrial and space experiments.

References

1 Einstein, A.: Über Gravitationswellen. In: Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften Berlin (1918), 154–167.

2 Einstein, A.: Näherungsweise Integration der Feldgleichungen der Gravitation. In: Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften Berlin (1916), 688–696.

3 Einstein, A., Rosen, N.: On Gravitational Waves. In: Journal of the Franklin Institute 223 (1937), 43–54.

Gravitational Waves: existence

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Russell Alan Hulse and Joseph Hooton Taylor Jr., working at the Arecibo Radio Observatory in Puerto Rico discovered a binary pulsar, i.e. two extremely dense and heavy stars in orbit around each other (1993 Nobel Prize in Physics).

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That was the case up until September 14, 2015, when the Laser Interferometer Gravitational-Wave Observatory (LIGO), for the first time, physically sensed distortions in space-time itself caused by passing gravitational waves generated by two colliding black holes nearly 1.3 billion light years away.

Einstein's field equations

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The solutions of the Einstein's field equations are metrics of a spacetime.

- A. Einstein (1915)

Mathematical models

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Definition

An exact plane fronted wave is a Lorentzian manifold $(\mathbb{R}^4, ds^2)$ endowed with the metric

$$ds^2 = dx_1^2 + dx_2^2 + 2\ du\ dv + H(x_1, x_2, u)\ du^2,$$

$(x_1, x_2, v, u) \in \mathbb{R}^4$, where $H : \mathbb{R}^3 \to \mathbb{R}$ is a non–null smooth function.

- H. Brinkmann, *Math. Ann.* **94** (1925)

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Scalar curvature: $S \equiv 0$ (always)
Ricci curvature: $\mathrm{Ric} \equiv 0 \iff \Delta_x H(x, u) \equiv 0$,
where $\Delta_x H$ denotes the Laplacian of H w.r.t. $x = (x_1, x_2)$.

Mathematical models: special waves

Definition

An exact (plane fronted) gravitational wave is an exact plane fronted wave such that the coefficient H has the special form

$$H(x_1, x_2, u) = f(u)(x_1^2 - x_2^2) + 2g(u)\ x_1 x_2$$

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Remark

$$H(x_1, x_2, u) \ = \begin{pmatrix} f(u) & g(u) \\ g(u) & -f(u) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Mathematical models: special waves

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Global properties of the exact models:

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Generalized Plane Waves

Definition

A Lorentzian manifold $(\mathcal{M}, \langle\cdot,\cdot\rangle_L)$ is called generalized plane wave, briefly *GPW*, if there exists a (connected) finite dimensional Riemannian manifold $(\mathcal{M}_0, \langle\cdot,\cdot\rangle_R)$ such that $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}^2$ and

$$\langle\cdot,\cdot\rangle_L \;=\; \langle\cdot,\cdot\rangle_R + 2dudv + \mathcal{H}(x,u)du^2,$$

where $x \in \mathcal{M}_0$, the variables (u,v) are the natural coordinates of $\mathbb{R}^2$ and the smooth function $\mathcal{H} : \mathcal{M}_0 \times \mathbb{R} \to \mathbb{R}$ is not identically zero.

- A.M. C., J.L. Flores, M. Sánchez, *Gen. Relativity Gravitation* **35** (2003)

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Definition

A Lorentzian manifold $(\mathcal{M}, \langle\cdot,\cdot\rangle_L)$ is called generalized plane wave, briefly *GPW*, if there exists a (connected) finite dimensional Riemannian manifold $(\mathcal{M}_0, \langle\cdot,\cdot\rangle_R)$ such that $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}^2$ and

$$\langle\cdot,\cdot\rangle_L \;=\; \langle\cdot,\cdot\rangle_R + 2dudv + \mathcal{H}(x,u)du^2,$$

where $x \in \mathcal{M}_0$, the variables (u, v) are the natural coordinates of $\mathbb{R}^2$ and the smooth function $\mathcal{H} : \mathcal{M}_0 \times \mathbb{R} \to \mathbb{R}$ is not identically zero.

- A.M. C., J.L. Flores, M. Sánchez, *Gen. Relativity Gravitation* **35** (2003)

$\mathcal{M}_0 = \mathbb{R}^2$ standard 2–dimensional Euclidean space

$\Longrightarrow$ GPW $\equiv$ exact plane fronted wave.

GPW: curvatures

GPW is a convenient generalization of exact plane fronted wave under both the physical and the mathematical viewpoint.

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GPW retains the characteristic structure of an exact plane fronted wave, but it drops additional symmetries.

Proposition

In a GPW:

(i) *the scalar curvature at each (x, v, u) is equal to the scalar curvature of the Riemannian part $(\mathcal{M}_0, \langle\cdot,\cdot\rangle_R)$ at x;*

(ii) *the Ricci tensor Ric is null if and only if the Riemannian Ricci tensor $Ric^{(R)}$ is null and $\Delta_x \mathcal{H} \equiv 0$.*

GPW: geodesics

Let $z :]a,b[\to \mathcal{M}$, $z(s) = (x(s), v(s), u(s))$ ($]a,b[\subseteq \mathbb{R}$), be a curve on $\mathcal{M}$ with constant energy $\langle \dot{z}(s), \dot{z}(s) \rangle_L = E_z$ for all $s \in]a,b[$. Assume $0 \in]a,b[$ and put $v_0 = v(0)$, $\Delta v = \dot{v}(0)$, $u_0 = u(0)$, $\Delta u = \dot{u}(0)$.

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- $u(s) = u_0 + s\Delta u$ for all $s \in]a,b[$;
- $x = x(s)$ is a solution of

$$D_s \dot{x} = -\nabla_x V_\Delta(x(s), s) \quad \text{for all } s \in]a,b[,$$

where $V_\Delta(x,s) = -\frac{(\Delta u)^2}{2}\, \mathcal{H}(x, u_0 + s\Delta u)$;

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  where $V_\Delta(x,s) = -\frac{(\Delta u)^2}{2}\mathcal{H}(x, u_0 + s\Delta u)$;
- either $\Delta u \neq 0$ and
  $$v(s) = v_0 + \frac{1}{2\Delta u}\int_0^s (E_z - \langle \dot{x}(\sigma), \dot{x}(\sigma)\rangle_R + 2V_\Delta(x(\sigma),\sigma))\, d\sigma$$
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for all $s \in]a,b[$,

or $\Delta u = 0$ and $v(s) = v_0 + s\Delta v$ for all $s \in]a,b[$.

GPW: geodesic connectedness

Proposition

A GPW $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}^2$ *is geodesically connected*

$$\Updownarrow$$

the problem

$$\begin{cases} D_s\dot{x}(s) = -\nabla_x V_\Delta(x(s), s) & \text{for all } s \in I \\ x(0) = x_0, \quad x(1) = x_1, \end{cases}$$

admits a solution for all $x_0, x_1 \in \mathcal{M}_0$*, all the possible values* $\Delta u \in \mathbb{R}$ *and all the initial points* $u_0 = u(0)$*.*

Points connected by a geodesic in a GPW

Let $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}^2$ be a GPW and fix $u_0, u_1 \in \mathbb{R}^2$, with $u_0 \leq u_1$. Suppose that $(\mathcal{M}_0, \langle \cdot, \cdot \rangle_R)$ is a complete Riemannian manifold.

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Theorem

There exist $p \in [0, 2[$, $\bar{x} \in \mathcal{M}_0$ and (positive) real numbers R_0, R_1, R_2 such that for all $(x, u) \in \mathcal{M}_0 \times [u_0, u_1]$ it is

$$\mathcal{H}(x, u) \geq -(R_0 d^2(x, \bar{x}) + R_1 d^p(x, \bar{x}) + R_2)$$

$$\Downarrow$$

two points $z_0 = (x_0, v_0, u_0), z_1 = (x_1, v_1, u_1) \in \mathcal{M}$ can be joined by a geodesic if $R_0(u_1 - u_0)^2 < \pi^2$.

- A.M. C., J.L. Flores, M. Sánchez, *JDE* **193** (2003)

GPW: geodesic connectedness

Corollary

Let $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}^2$ *be a GPW such that* $(\mathcal{M}_0, \langle\cdot,\cdot\rangle_R)$ *is a complete Riemannian manifold.*

There exist $\bar{x} \in \mathcal{M}_0$, *(positive) continuous functions* $R_1(u), R_2(u), p(u)$, *with* $p(u) < 2$, *such that*

$\mathcal{H}(x,u) \geq -(R_1(u)d^{p(u)}(x,\bar{x}) + R_2(u))$ *for all* $(x,u) \in \mathcal{M}_0 \times \mathbb{R}$

$$\Downarrow$$

$\mathcal{M}$ *is geodesically connected.*

GPW: global hyperbolicity

Definition

A spacetime is globally hyperbolic if there exists a (smooth) spacelike Cauchy hypersurface, i.e., a subset which is crossed exactly once by any inextendible timelike curve.

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Theorem

Let $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}^2$ be a GPW with $(\mathcal{M}_0, \langle\cdot,\cdot\rangle_R)$ complete. Assume that $\bar{x} \in \mathcal{M}_0$, (positive) continuous functions $R_1(u), R_2(u), p(u)$, with $p(u) < 2$, exist such that

$\mathcal{H}(x,u) \geq -(R_1(u) d^{p(u)}(x,\bar{x}) + R_2(u))$ *for all* $(x,u) \in \mathcal{M}_0 \times \mathbb{R}$

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$\mathcal{M}$ is globally hyperbolic.

- J.L. Flores, M. Sánchez, *Class. Quant. Grav.* **20** (2003)

Global hyperbolicity and geodetic connectedness

Remark

Same assumptions imply both a GPW is geodesically connected and it is globally hyperbolic.

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Theorem

Let $\mathcal{M}$ be a globally hyperbolic GPW with a complete Cauchy hypersurface

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Same assumptions imply both a GPW is geodesically connected and it is globally hyperbolic.

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$K = \partial v$ is a complete lightlike Killing vector field on $\mathcal{M}$.

GPW: geodesic completeness

Theorem

A GPW $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}^2$ *is geodesically complete*

$$\Updownarrow$$

- $(\mathcal{M}_0, \langle \cdot, \cdot \rangle_R)$ *is a complete Riemannian manifold,*
- *the trajectories of*

$$D_s \dot{x} = -\nabla_x V_\Delta(x, s)$$

are complete, i.e., each of them can be extended so as to be defined on all $\mathbb{R}$.

Positively complete functions

It is enough studying the behavior of the Riemannian trajectories which are solutions of the equation

$D_s\dot{x} = -\nabla_x V_\Delta(x,s)$ with $V_\Delta(x,s) = -\frac{(\Delta u)^2}{2}\mathcal{H}(x, u_0 + s\Delta u)$.

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Definition

A function $U_0 : [0,+\infty[\to \mathbb{R}$ is positively complete if it is a nonincreasing C^2 map such that

$$\int_0^{+\infty} \frac{dt}{\sqrt{\alpha - U_0(t)}} = +\infty,$$

for some (and thus any) $\alpha > U_0(0) = \sup U_0([0,+\infty[)$.

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Example

$U_0(t) = -R_0 t^p$ with $R_0 > 0$ and $0 \leq p \leq 2$ is positively complete.

Autonomous case

Let $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}^2$ be a GPW such that $(\mathcal{M}_0, \langle\cdot,\cdot\rangle_R)$ is complete and $\mathcal{H}$ is autonomous, i.e., $\mathcal{H}(x,u) \equiv \mathcal{H}(x)$.

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Proposition

There exist $r > 0$, $\bar{x} \in \mathcal{M}_0$ and a positively complete function $U_0 : [0,+\infty[\to \mathbb{R}$ such that
$\mathcal{H}(x) \leq -U_0(d(x,\bar{x}))$ for all $x \in \mathcal{M}_0$ such that $d(x,\bar{x}) \geq r$,
$\Longrightarrow$ $\mathcal{M}$ is geodesically complete.

- A. Weinstein, J. Marsden, *Proc. AMS* **26** (1970)

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Non–autonomous case

Let $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}^2$ be a GPW such that $(\mathcal{M}_0, \langle \cdot, \cdot \rangle)$ is a complete Riemannian manifold and $\mathcal{H} : \mathcal{M}_0 \times \mathbb{R} \to \mathbb{R}$ is a smooth function.

Proposition

There exist two continuous functions α_0, $\beta_0 : \mathbb{R} \to \mathbb{R}$ *such that*
$\mathcal{H}(x, u) \le \beta_0(u)$ *and* $\left|\frac{\partial \mathcal{H}}{\partial u}(x, u)\right| \le \alpha_0(u)\,(\beta_0(u) - \mathcal{H}(x, u))$
for all $(x, u) \in \mathcal{M}_0 \times \mathbb{R}$, $\Longrightarrow$ $(\mathcal{M}, g)$ *is geodesically complete.*

- A.M. C., A. Romero, M. Sánchez, In: *Proc. Int. Meeting on Differential Geometry*, Córdoba, 2012

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Proposition

$\nabla^{\mathcal{M}}\mathcal{H}$ *grows at most linearly in* $\mathcal{M}$ *along finite times*
$\Longrightarrow$ $(\mathcal{M}, g)$ *is geodesically complete.*

- A.M. C., A. Romero, M. Sánchez, *ARMA* **208** (2013)