CURVES IN LORENTZ-MINKOWSKI PLANE WITH PRESCRIBED CURVATURE

Ildefonso Castro
(Ildefonso Castro-Infantes and Jesús Castro-Infantes)

Universidad de Jaén

ugr Universidad de Granada

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Fundamental Theorem for plane curves

THEOREM

Prescribe $\kappa = \kappa(s)$:

$\theta(s) = \int \kappa(s)\, ds, \quad x(s) = \int \cos\theta(s)\, ds, \quad y(s) = \int \sin\theta(s)\, ds$

$\Rightarrow$ $(x(s), y(s))$ unique up to rigid motions

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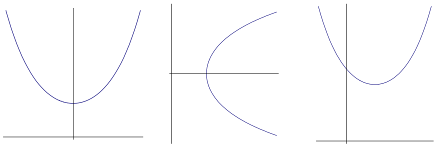
Example (Catenary)

$\kappa(s) = \frac{1}{1+s^2} \Rightarrow \theta(s) = \arctan s$

$x(s) = \log\left(s + \sqrt{s^2+1}\right),\ y(s) = \sqrt{1+s^2} \leftrightarrow y = \cosh x$



Singer's Problem

[D. Singer: *Curves whose curvature depends on distance from the origin.* Amer. Math. Monthly **106** (1999), 835–841.]

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Can a plane curve be determined if its curvature is given in terms of its position?

$$\kappa = \kappa(x, y), \quad \frac{x'(t)y''(t) - y'(t)x''(t)}{\left(x'(t)^2 + y'(t)^2\right)^{3/2}} = \kappa(x(t), y(t))$$

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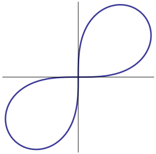
Bernoulli lemniscate: $r^2 = 3 \sin 2\theta$



Euler elastic curves

Elastica under *tension* $\sigma \in \mathbb{R}$:
Critical points of $\int (\kappa^2 + \sigma) ds$: $2\ddot{\kappa} + \kappa^3 - \sigma\kappa = 0$

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$$\kappa(y)=2\lambda y,\ \lambda>0$$

$$\int \kappa(y)dy=\lambda y^2+c$$

Tension $\sigma=-4\lambda c$

Maximum curvature $k_0=2\sqrt{\lambda}\sqrt{1-c}$, $c<1$

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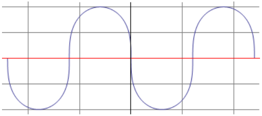
- $c > -1$, *wavelike*:

$\kappa(s) = k_0 \operatorname{cn}\left(\frac{k_0 s}{2p}, p\right)$,
$p^2 = \frac{1-c}{2}$, $s \in \mathbb{R}$



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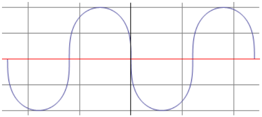

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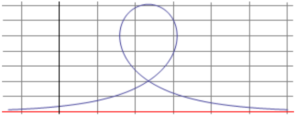

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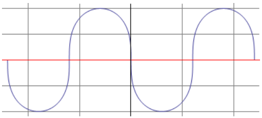

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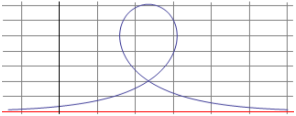

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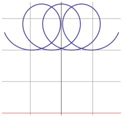

Plane curves with prescribed curvature I

$$\kappa(x, y) = \kappa(y)$$

[I. Castro and I. Castro-Infantes: *Plane curves with curvature depending on distance to a line*. Diff. Geom. Appl. **44** (2016), 77–97.]

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Theorem $\kappa = \kappa(y)$

Prescribe $\kappa = \kappa(y)$ continuous.
The problem of determining a curve $\gamma(s) = (x(s), y(s))$ -s arc length- with curvature $\kappa(y)$ is solvable by three quadratures:

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3. $x(s) = -\int \mathcal{K}(y(s))ds$.

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- γ is uniquely determined, up to translations in x-direction, by $\mathcal{K}(y)$.

Uniqueness results for plane curves I

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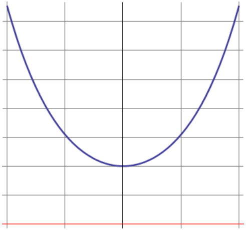
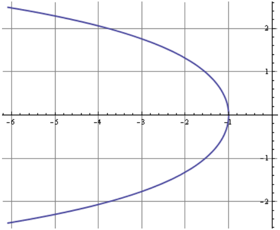
The **catenary** $y = \cosh x$, $x \in \mathbb{R}$,
is the only plane curve (up to x-translations)
with geometric linear momentum $\mathcal{K}(y) = -1/y$
(and curvature $\boxed{\kappa(y) = 1/y^2}$).

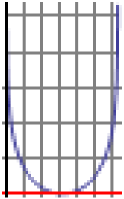


The **catenary** $x = -\cosh y$, $y \in \mathbb{R}$,
is the only plane curve (up to x-translations)
with geometric linear momentum $\mathcal{K}(y) = \tanh y$
(and curvature $\boxed{\kappa(y) = 1/\cosh^2 y}$).

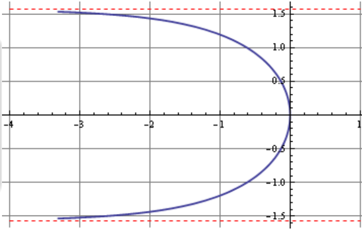


Uniqueness results for plane curves II

The **grim-reaper** $y = -\log \sin x$, $0 < x < \pi$, is the only plane curve (up to x-translations) with geometric linear momentum $\mathcal{K}(y) = -e^{-y}$ (and curvature $\boxed{\kappa(y) = e^{-y}}$).



The **grim-reaper** $x = \log \cos y$, $y \in (-\frac{\pi}{2}, \frac{\pi}{2})$, is the only plane curve (up to x-translations) with geometric linear momentum $\mathcal{K}(y) = \sin y$ (and curvature $\boxed{\kappa(y) = \cos y}$).



Plane curves with prescribed curvature II

$$\kappa(x, y) = \kappa(\sqrt{x^2 + y^2})$$

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Theorem $\kappa = \kappa(r)$

Prescribe $\kappa = \kappa(r)$ such that $r\kappa(r)$ continuous.
The problem of determining a curve $\gamma(s) = r(s)\, e^{i\theta(s)}$ -s arc length- with curvature $\kappa(r)$ is solvable by three quadratures:

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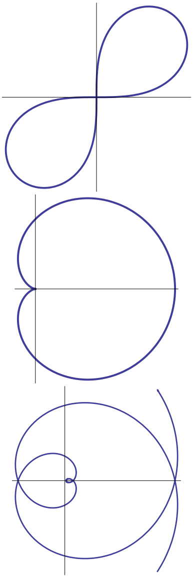
- γ is uniquely determined, up to rotations, by $\mathcal{K}(r)$.

Uniqueness results for plane curves III

The **Bernoulli lemniscate** $r^2 = 3\sin 2\theta$
is the only plane curve (up to rotations)
with geometric angular momentum $\mathcal{K}(r) = r^3/3$
(and curvature $\boxed{\kappa(r) = r}$).

The **cardioid** $r = \frac{1}{2}(1 + \cos\theta)$
is the only plane curve (up to rotations)
with geometric angular momentum $\mathcal{K}(r) = r\sqrt{r}$
(and curvature $\boxed{\kappa(r) = \frac{3}{2\sqrt{r}}}$).

The **Norwich spiral** is the only (non circular)
plane curve (up to rotations) with curvature
$\boxed{\kappa(r) = 1/r}$.



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• $\gamma = (x, y) : I \subseteq \mathbb{R} \to \mathbb{R}^2$ spacelike (resp. timelike) if $\gamma'(t)$ spacelike (resp. timelike) $\forall t \in I$; $\gamma(t_0)$ lightlike point if $\gamma'(t_0)$ lightlike vector

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• $\gamma = (x, y)$ *unit-speed* spacelike (resp. timelike) if $g(\dot{\gamma}(s), \dot{\gamma}(s)) = \epsilon$, $\forall s \in I$ ($\epsilon = 1$ if γ spacelike, $\epsilon = -1$ if γ timelike)

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• $\gamma = (x, y)$ *unit-speed* spacelike (resp. timelike) if $g(\dot\gamma(s), \dot\gamma(s)) = \epsilon$, $\forall s \in I$ ($\epsilon = 1$ if γ spacelike, $\epsilon = -1$ if γ timelike)

• $T = \dot\gamma = (\dot x, \dot y)$, $N = \dot\gamma^\perp = (\dot y, \dot x)$, $g(T, T) = \epsilon$, $g(N, N) = -\epsilon$

Frenet frame and eqns: $\dot T(s) = \kappa(s) N(s)$, $\dot N(s) = \kappa(s) T(s)$

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$$\mathbb{L}^2 := (\mathbb{R}^2, g = -dx^2 + dy^2)$$

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• $T = \dot\gamma = (\dot x, \dot y)$, $N = \dot\gamma^{\perp} = (\dot y, \dot x)$, $g(T,T) = \epsilon$, $g(N,N) = -\epsilon$
Frenet frame and eqns: $\dot T(s) = \kappa(s) N(s)$, $\dot N(s) = \kappa(s) T(s)$

Theorem

Prescribe $\kappa = \kappa(s)$:
Any *spacelike* curve $\alpha(s)$ in $\mathbb{L}^2$ can be represented (up to isometries) by

$$\alpha(s) = \left(\int \sinh \varphi(s) ds, \int \cosh \varphi(s) ds \right) \text{ with } \varphi(s) = \int \kappa(s) ds.$$

Any *timelike* curve $\beta(s)$ in $\mathbb{L}^2$ can be represented (up to isometries) by

$$\beta(s) = \left(\int \cosh \phi(s) ds, \int \sinh \phi(s) ds \right) \text{ with } \phi(s) = \int \kappa(s) ds.$$

Curves in Lorentz-Minkowski plane

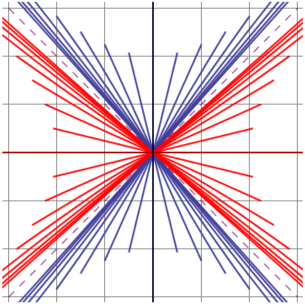
Geodesics

The *spacelike* geodesics can be written as:

$$\alpha_{\varphi_0}(s) = (\sinh \varphi_0\, s, \cosh \varphi_0\, s),\ s \in \mathbb{R},\ \ \varphi_0 \in \mathbb{R},$$

and the *timelike* geodesics can be written as:

$$\beta_{\phi_0}(s) = (\cosh \phi_0\, s, \sinh \phi_0\, s),\ s \in \mathbb{R},\ \ \phi_0 \in \mathbb{R}.$$



Lorentzian Pseudodistance

Define the *Lorentzian pseudodistance* by

$$\delta : \mathbb{L}^2 \times \mathbb{L}^2 \to [0, +\infty), \; \delta(P, Q) = \sqrt{|g(\overrightarrow{PQ}, \overrightarrow{PQ})|}$$

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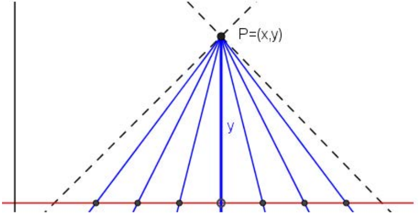
Fix the timelike geodesic β_0, i.e. the x-axis.



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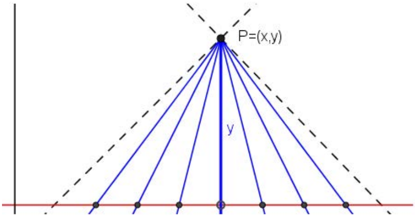

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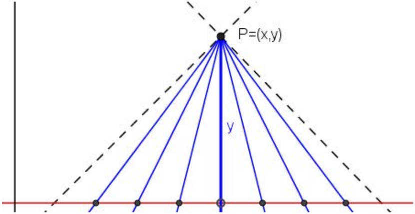

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$|y|$: maximum Lorentzian pseudodistance through spacelike geodesics from $P = (x, y)$, $y \neq 0$, to the x-axis.

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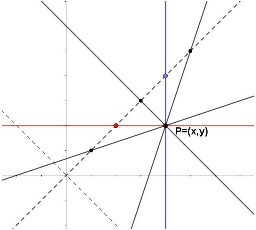

$P = (x, y) \in \mathbb{L}^2$, $x \neq y$; spacelike and timelike geodesics γ_m, $m \in \mathbb{R} \cup \{\infty\}$, $m \neq 1$; $P' = \left(\frac{mx-y}{m-1}, \frac{mx-y}{m-1}\right)$

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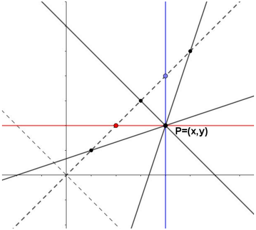

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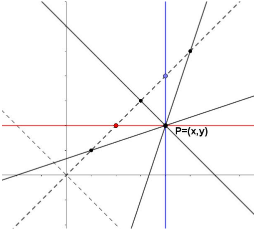

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$|y - x|$: Lorentzian pseudodistance from $P = (x, y) \in \mathbb{L}^2$, $x \neq y$, to the lightlike geodesic $x = y$ through the horizontal timelike geodesic or the vertical spacelike geodesic.

Singer's Problem in $\mathbb{L}^2$

Determine those (spacelike and timelike) curves $\gamma = (x, y)$ in $\mathbb{L}^2$ whose curvature κ depends on some given function $\kappa = \kappa(x, y)$

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$\kappa(x, y) = \kappa(y)$

Theorem

Prescribe $\kappa = \kappa(y)$ continuous.
Then the problem of determining a spacelike or timelike curve $(x(s), y(s))$ -s arc length- is solvable by three quadratures ($\epsilon = 1$ spacelike, $\epsilon = -1$ timelike):

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▶ Such a curve is uniquely determined by $\mathcal{K}(y)$ up to a x-translation.
• $\mathcal{K}(y)$ will distinguish geometrically the curves inside a same family by their relative position with respect to the x-axis.

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Geodesics: $\kappa \equiv 0$

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- $\mathcal{K}(y) = c \in \mathbb{R}$. $s = \int \frac{dy}{\sqrt{c^2+\epsilon}} = \frac{y}{\sqrt{c^2+\epsilon}}$, $c^2+\epsilon > 0$.
$$x(s) = c\,s,\ y(s) = \sqrt{c^2+\epsilon}\,s,\ s \in \mathbb{R}.$$

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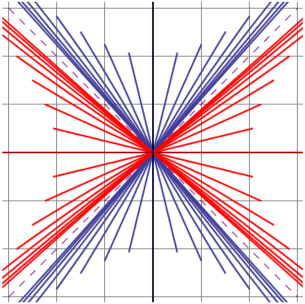
$\epsilon = 1$: $K \equiv c := \sinh \varphi_0 \rightarrow$ spacelike geodesics α_{φ_0}.
$c = 0 = \varphi_0$ corresponds to the y-axis.

$\epsilon = -1$: $K \equiv c := \cosh \phi_0 \rightarrow$ timelike geodesics β_{ϕ_0}.
$c = 1 \Leftrightarrow \phi_0 = 0$ corresponds to the x-axis.



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- $\mathcal{K}(y) = k_0 y + c,\ c \in \mathbb{R}$. $\quad s = \int \frac{dy}{\sqrt{(k_0 y + c)^2 + \epsilon}}$.

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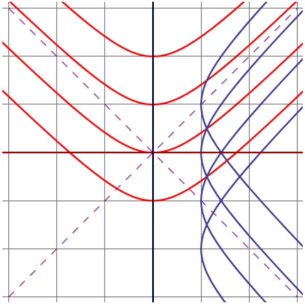
$\epsilon = 1$: $s = \mathrm{arcsinh}(k_0 y + c)/k_0$.
$x(s) = \cosh(k_0 s)/k_0$, $y(s) = (\sinh(k_0 s) - c)/k_0$, $s \in \mathbb{R}$.
$\epsilon = -1$: $s = \mathrm{arccosh}(k_0 y + c)/k_0$
$x(s) = \sinh(k_0 s)/k_0$, $y(s) = (\cosh(k_0 s) - c)/k_0$, $s \in \mathbb{R}$.



Spacelike and timelike pseudocircles in $\mathbb{L}^2$ of radius $1/k_0$.

Elasticae in $\mathbb{L}^2$

Definition

γ, spacelike or timelike curve in $\mathbb{L}^2$, *elastica under tension* σ if $2\ddot{\kappa} - \kappa^3 - \sigma\kappa = 0$, $\sigma \in \mathbb{R}$. *Energy* $E := \dot{\kappa}^2 - \frac{1}{4}\kappa^4 - \frac{\sigma}{2}\kappa^2$.

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Proposition

γ spacelike or timelike curve in $\mathbb{L}^2$

(i) If $\kappa(y) = 2ay + b$, $a \neq 0$, $b \in \mathbb{R}$, and $\mathcal{K}(y) = ay^2 + by + c$, $a \neq 0$, $b, c \in \mathbb{R}$, then γ elastica under tension $\sigma = 4ac - b^2$ and energy $E = 4\epsilon a^2 + \sigma^2/4$ (where $\epsilon = 1$ if γ is spacelike and $\epsilon = -1$ if γ is timelike).

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(ii) If γ elastica under tension σ and energy E, with $E \neq \sigma^2/4$, then $\kappa(y) = 2ay + b$, $a \neq 0$, $b \in \mathbb{R}$.

Spacelike elasticae: $\boxed{\kappa(y) = 2y,\ \epsilon = 1}$

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- $\mathcal{K}(y) = y^2 + c,\ c = \sinh\eta \in \mathbb{R}$ $\quad (s_\eta = \sinh\eta$ and $c_\eta = \cosh\eta)$

$$x_\eta(s) = (s_\eta + c_\eta)s + \sqrt{c_\eta}\left(\mathrm{cn}(\sqrt{c_\eta}\, s, k_\eta)\left(k_\eta^2\, \mathrm{sd}(\sqrt{c_\eta}\, s, k_\eta) - \mathrm{ds}(\sqrt{c_\eta}\, s, k_\eta)\right) - 2E(\sqrt{c_\eta}\, s, k_\eta)\right)$$

$$y_\eta(s) = \sqrt{c_\eta}\ \mathrm{cs}(\sqrt{c_\eta}\, s, k_\eta)\, \mathrm{nd}(\sqrt{c_\eta}\, s, k_\eta),\ \ k_\eta^2 = \frac{1-\tanh\eta}{2}$$

$$s \in (2mK(k_\eta)/\sqrt{c}_\eta, 2(m+1)K(k_\eta)/\sqrt{c}_\eta),\ m \in \mathbb{N}$$

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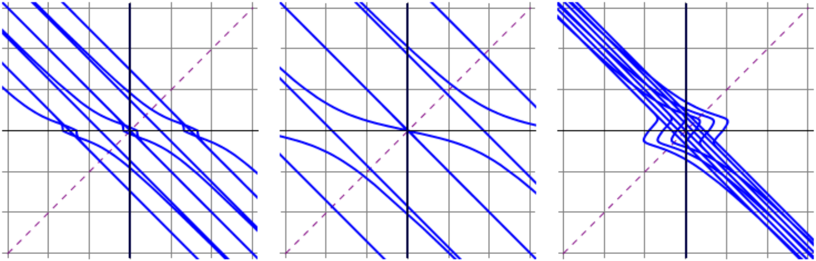

Spacelike elastic curves $\alpha_\eta = (x_\eta, y_\eta)$, $(\eta = 0, 1{,}5, -1{,}5)$.

Timelike elasticae: $\boxed{\kappa(y) = 2y,\ \epsilon = -1}$

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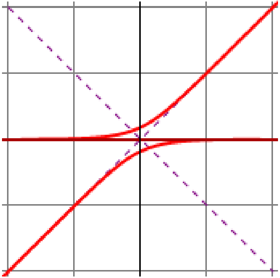
- $\mathcal{K}(y) = y^2 + 1$ $(c = 1)$.

$x_1(s) = s - \sqrt{2}\coth(\sqrt{2}s),$

$y_1(s) = -\frac{\sqrt{2}}{\sinh(\sqrt{2}s)},\ \ s \neq 0.$

$\kappa_1(s) = -\frac{2\sqrt{2}}{\sinh(\sqrt{2}s)}.$

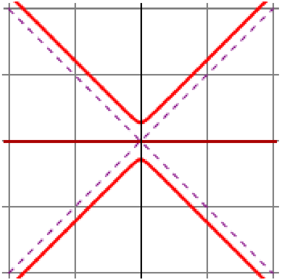


- $\mathcal{K}(y) = y^2 - 1$ $(c = -1)$.

$x_{-1}(s) = \sqrt{2}\tan(\sqrt{2}s) - s,$

$y_{-1}(s) = \pm\frac{\sqrt{2}}{\cos(\sqrt{2}s)},\ \ |s| < \frac{\pi}{2\sqrt{2}}.$

$\kappa_{-1}(s) = \frac{\mp 2\sqrt{2}}{\cos(\sqrt{2}s)}.$



Timelike elasticae: $\boxed{\kappa(y) = 2y,\ \epsilon = -1}$

- $\mathcal{K}(y) = y^2 + \cosh^2\delta$, $\delta > 0$ ($c > 1$)

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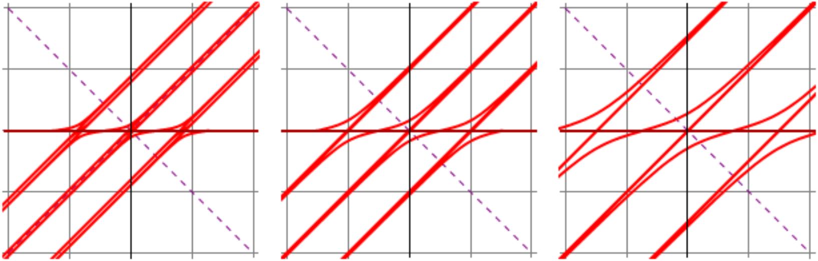
$x_\delta(s) = c_\delta^2 s + \sqrt{c_\delta^2+1}\left(\mathrm{dn}(\sqrt{c_\delta^2+1}\,s, k_\delta)\,\mathrm{tn}(\sqrt{c_\delta^2+1}\,s, k_\delta) - E(\sqrt{c_\delta^2+1}\,s, k_\delta)\right),$

$y_\delta(s) = s_\delta\ \mathrm{tn}(\sqrt{c_\delta^2+1}\,s, k_\delta),\ \ k_\delta^2 = \frac{2}{1+\cosh^2\delta},$

$s \in \left((2m-1)K(k_\delta)/\sqrt{c_\delta^2+1}, (2m+1)K(k_\delta)/\sqrt{c_\delta^2+1}\right),\ m \in \mathbb{N}.$

$\kappa_\delta(s) = 2s_\delta\ \mathrm{tn}(\sqrt{c_\delta^2+1}\,s, k_\delta).$



Timelike elastic curves $\beta_\delta = (x_\delta, y_\delta)$ $(\delta = 0{,}5, 1, 1{,}5)$.

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$x_\psi(s) = s + \sqrt{2}\left(\mathrm{dn}(\sqrt{2}\,s, k_\psi)\,\mathrm{tn}(\sqrt{2}\,s, k_\psi) - E(\sqrt{2}\,s, k_\psi)\right),$

$y_\psi(s) = \sqrt{1 - s_\psi}\,\mathrm{nc}(\sqrt{2}\,s, k_\psi),\ \ k_\psi^2 = \frac{1+\sin\psi}{2},$

$s \in \left((2m-1)K(k_\psi)/\sqrt{2}, (2m+1)K(k_\psi)/\sqrt{2}\right),\ m \in \mathbb{N}.$

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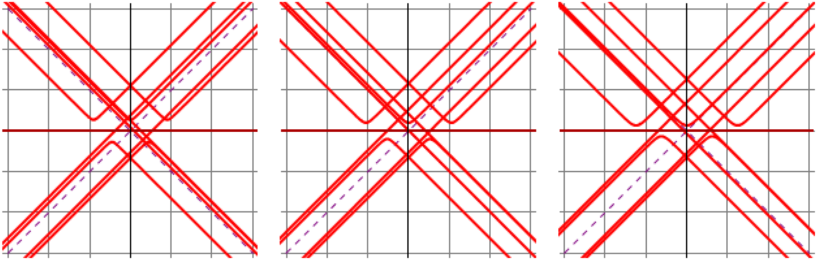

Timelike elastic curves $\beta_\psi = (x_\psi, y_\psi)$ $(\psi = -\pi/4, 0, \pi/6)$.

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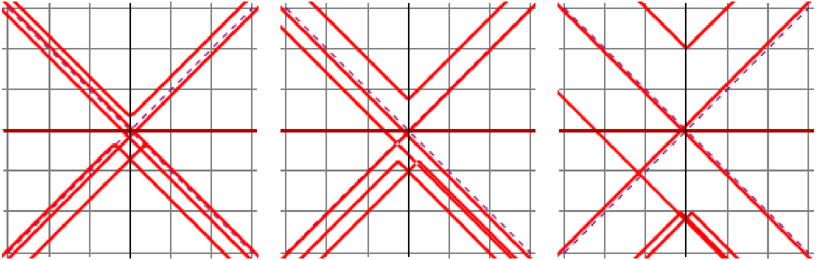
- $\mathcal{K}(y) = y^2 - \cosh^2\tau,\ \tau > 0$, $(c < -1)$

$x_\tau(s) = = s + \sqrt{1+c_\tau^2}\left(\mathrm{dn}(\sqrt{1+c_\tau^2}\, s, k_\tau)\,\mathrm{tn}(\sqrt{1+c_\tau^2}\, s, k_\tau) - E(\sqrt{1+c_\tau^2}\, s, k_\tau)\right)$,

$y_\tau(s) = \sqrt{1+c_\tau^2}\ \mathrm{dc}(\sqrt{1+c_\tau^2}\, s, k_\tau)$, $k_\tau^2 = \frac{\sinh^2\tau}{1+\cosh^2\tau}$,

$s \in \left((2m-1)K(k_\tau)/\sqrt{1+c_\tau^2}, (2m+1)K(k_\tau)/\sqrt{1+c_\tau^2}\right)$, $m \in \mathbb{N}$.

$\kappa_\tau(s) = 2\sqrt{1+c_\tau^2}\ \mathrm{dc}(\sqrt{1+c_\tau^2}\, s, k_\tau)$.



Timelike elastic curves $\beta_\tau = (x_\tau, y_\tau)$, $(\tau = 1, 2, 3)$.

$\kappa(y) = 1/y^2$

$\kappa(y) = 1/y^2$

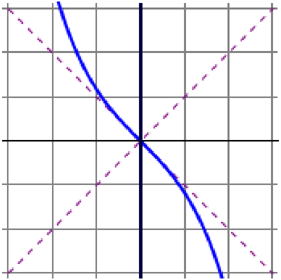
- $\mathcal{K}(y) = -1/y$

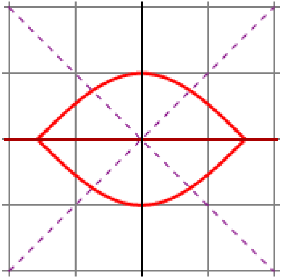
$\epsilon = 1$. Spacelike case:
$x(s) = \mp \operatorname{arccosh} s$, $s > 1$.
$y(s) = \pm\sqrt{s^2 - 1}$, $|s| > 1$.
$\kappa(s) = \frac{1}{s^2-1}$, $s > 1$.

$y = -\sinh x$, $x \in \mathbb{R}$.

$\epsilon = -1$. Timelike case:
$x(s) = \mp \arcsin s$, $|s| < 1$.
$y(s) = \pm\sqrt{1 - s^2}$, $|s| < 1$.
$\kappa(s) = \frac{1}{1-s^2}$, $|s| < 1$.

$y = \pm \cos x$, $|x| < \pi/2$.





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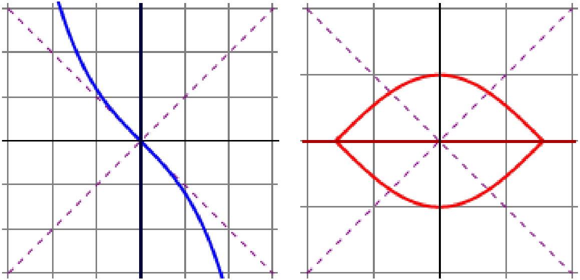
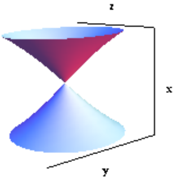
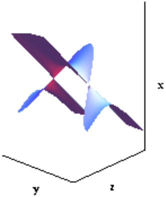


"Lorentzian catenaries"

Lorentzian catenaries.

Kobayashi introduced in 1993, studying maximal rotation surfaces in $\mathbb{L}^3$, the catenoid of the first kind with equation $y^2 + z^2 - \sinh^2 x = 0$ and the catenoid of the second kind with equation $x^2 - z^2 = \cos^2 y$.





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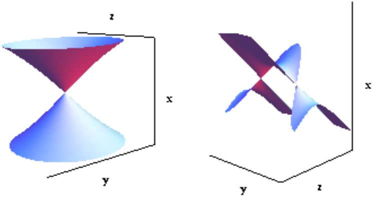

The generatrix curves of both catenoids coincide with the graph $y = -\sinh x$, $x \in \mathbb{R}$ and the bigraph $x = \pm \cos y$, $|y| < \pi/2$.

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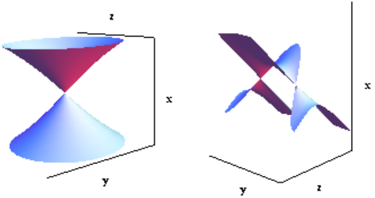

The generatrix curves of both catenoids coincide with the graph $y = -\sinh x$, $x \in \mathbb{R}$ and the bigraph $x = \pm \cos y$, $|y| < \pi/2$.

1. The Lorentzian catenary of the first kind $y = -\sinh x$, $x \in \mathbb{R}$, is the only *spacelike* curve (up to translations in the x-direction) with geometric linear momentum $\mathcal{K}(y) = -1/y$.
2. The Lorentzian catenary of the second kind $x = \pm \cos y$, $|y| < \pi/2$, is the only *spacelike* curve (up to translations in the y-direction) with geometric linear momentum $\mathcal{K}(x) = -1/x$.

$\kappa(y) = 1/y^2$

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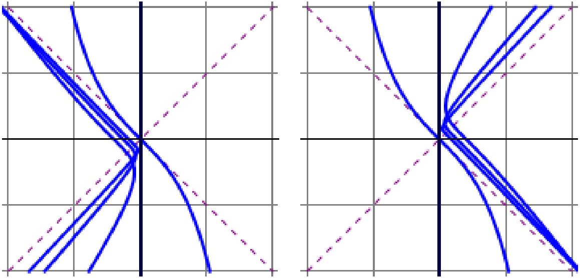
- $\mathcal{K}(y) = c - 1/y,\ c \neq 0$. $\epsilon = 1$, Spacelike case:

$$x = \frac{1}{c^2+1}\left(c\sqrt{(c^2+1)y^2 - 2cy + 1} - \frac{1}{\sqrt{c^2+1}}\,\mathrm{arcsinh}((c^2+1)y - c)\right).$$



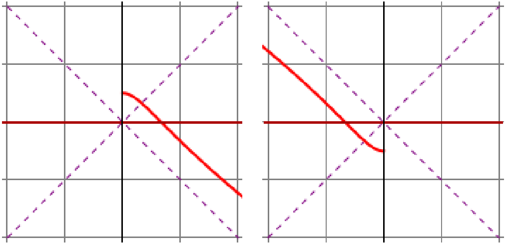
Curves with $\mathcal{K}(y) = c - 1/y$; $c \leq 0$ (left) and $c \geq 0$ (right).

$\kappa(y) = 1/y^2$

- $\mathcal{K}(y) = c - 1/y,\ c \neq 0.$ $\epsilon = -1$, Timelike case:

$\cdot\ \mathcal{K}(y) = 1 - 1/y$:
$x = \frac{(2-y)\sqrt{1-2y}}{3},\ y < 1/2.$

$\cdot\ \mathcal{K}(y) = -1 - 1/y$:
$x = -\frac{(2+y)\sqrt{1+2y}}{3},\ y > -1/2.$



$\cdot\ \mathcal{K}(y) = c - 1/y,\ |c| > 1$:

$$x = \frac{1}{c^2-1}\left(c\sqrt{(c^2-1)y^2 - 2cy + 1} + \frac{\log\left(2(\sqrt{c^2-1}\sqrt{(c^2-1)y^2-2cy+1}+(c^2-1)y-c)\right)}{\sqrt{c^2-1}}\right).$$

$\cdot\ \mathcal{K}(y) = c - 1/y,\ |c| < 1$:

$$x = \frac{1}{c^2-1}\left(c\sqrt{(c^2-1)y^2 - 2cy + 1} - \frac{1}{\sqrt{1-c^2}}\arcsin((c^2-1)y - c)\right)$$

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$x(s) =$
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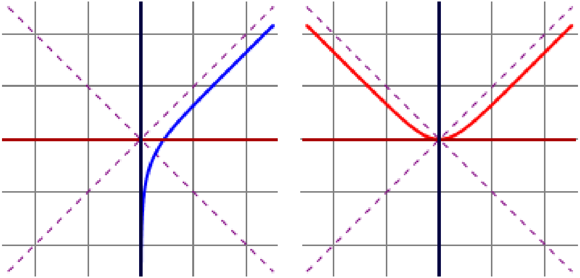
$y = \log(\cosh x),\ x \in \mathbb{R}.$



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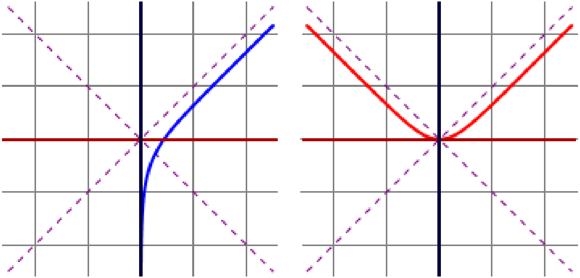
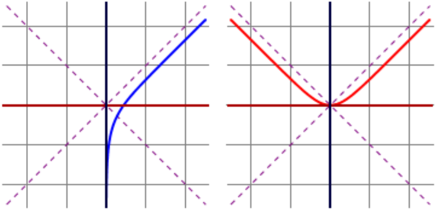


"Lorentzian grim-reapers"

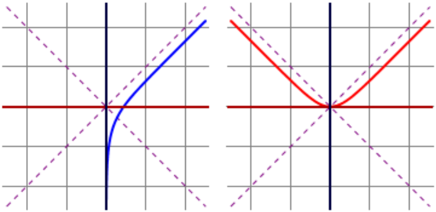
Lorentzian grim-reapers



Both curves satisfy the translating-type soliton equation:

$$\kappa = g((0,1), N)$$

Lorentzian grim-reapers



Both curves satisfy the translating-type soliton equation:

$$\kappa = g((0,1), N)$$

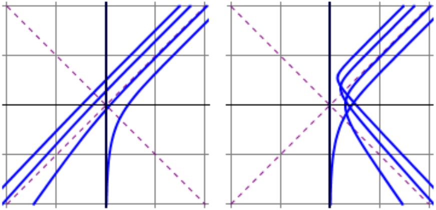
1. The Lorentzian grim-reaper $y = \log(\sinh x)$, $x > 0$, is the only *spacelike* curve (up to x-translations) in $\mathbb{L}^2$ with geometric linear momentum $\mathcal{K}(y) = e^y$.
2. The Lorentzian grim-reaper $y = \log(\cosh x)$, $x \in \mathbb{R}$, is the only *timelike* curve (up to x-translations) in $\mathbb{L}^2$ with geometric linear momentum $\mathcal{K}(y) = e^y$.

$\kappa(y) = e^y$

- $\mathcal{K}(y) = e^y + c,\ c \neq 0.$

Spacelike case ($\epsilon = 1$):

$$x = \operatorname{arcsinh}(e^y + c) - \frac{c}{\sqrt{c^2+1}} \operatorname{arcsinh}\left(c + (c^2+1)e^{-y}\right).$$



Timelike case ($\epsilon = -1$):

$\cdot\mathcal{K}(y) = e^y + 1$:

$$x = 2\log(\sqrt{e^y} + \sqrt{e^y + 2}) - \sqrt{1 + 2e^{-y}}.$$

$\cdot\mathcal{K}(y) = e^y + c,\ |c| > 1$:

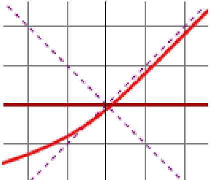
$$x = \log\left(2(\sqrt{P(e^y)} + e^y + c)\right) - \frac{c\log\left(2e^{-y}(\sqrt{c^2-1}\sqrt{P(e^y)} + ce^y + c^2 - 1)\right)}{\sqrt{c^2-1}}$$

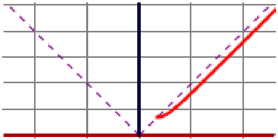
$\cdot\mathcal{K}(y) = e^y - 1$:

$$x = 2\log(\sqrt{e^y} + \sqrt{e^y - 2}) - \sqrt{1 - 2e^{-y}}.$$

$\cdot\mathcal{K}(y) = e^y + c,\ |c| < 1$:

$$x = \log\left(2(\sqrt{P(e^y)} + e^y + c)\right) + \frac{c}{\sqrt{1-c^2}} \arcsin\left(c + (c^2-1)e^{-y}\right).$$





Index

$$\kappa(x, y) = \kappa(v),\ v := y - x$$

Theorem

Prescribe $\kappa=\kappa(v)$ continuous.
Then the problem of determining a spacelike or timelike curve

$$\left(\frac{u(s)-v(s)}{2},\frac{u(s)+v(s)}{2}\right)$$

-s arc length- is solvable by three quadratures
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▶ Such a curve is uniquely determined by $\mathcal{K}(v)$ up to a u-translation.
• $\mathcal{K}(v)$ will distinguish geometrically the curves inside a same family by their relative position with respect to the u-axis.

Examples: constant curvature

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Geodesics: $\kappa \equiv 0$

• $\mathcal{K}(v) = -\epsilon/c$, $c \neq 0$. $u(s) = -\epsilon s/c$, $v(s) = -cs$, $s \in \mathbb{R}$
(lines passing through the origin with slope $m = \frac{\epsilon + c^2}{\epsilon - c^2}$).
$\epsilon = 1 \Rightarrow |m| > 1$ spacelike geodesics, $\epsilon = -1 \Rightarrow |m| < 1$ timelike geodesics.

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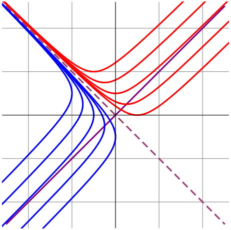
Circles: $\kappa \equiv k_0 > 0$

• $\mathcal{K}(v) = \frac{-\epsilon}{c + k_0 v}$, $c \in \mathbb{R}$. $u(s) = -\epsilon e^{k_0 s}/k_0$, $v(s) = (e^{-k_0 s} - c)/k_0$.
$\epsilon = 1 \Rightarrow x(s) = (-\cosh(k_0 s) + c/2)/k_0$, $y(s) = -(\sinh(k_0 s) + c/2)/k_0$.
$\epsilon = -1 \Rightarrow x(s) = (\sinh(k_0 s) + c/2)/k_0$, $y(s) = (\cosh(k_0 s) - c/2)/k_0$.



Spacelike and timelike pseudocircles in $\mathbb{L}^2$ of radius $1/k_0$.

$\kappa(v) = 2v$

σ-elastica: $2\ddot{\kappa} - \kappa^3 - \sigma\kappa = 0$, $\sigma \in \mathbb{R}$.

Energy $E \in \mathbb{R}$: $E = \dot{\kappa}^2 - \frac{1}{4}\kappa^4 - \frac{\sigma}{2}\kappa^2$.

$\boxed{E = \sigma^2/4\ ?}$

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- $\mathcal{K}(v) = -\frac{\epsilon}{v^2+c}$, $c \in \mathbb{R}$ ($\epsilon = 1$ spacelike, $\epsilon = -1$ timelike).

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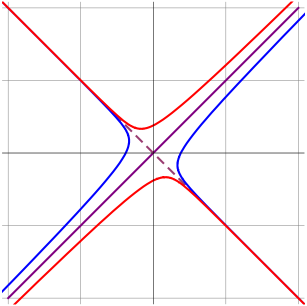
- $\mathcal{K}(v) = -\frac{\epsilon}{v^2+c}$, $c \in \mathbb{R}$ ($\epsilon = 1$ spacelike, $\epsilon = -1$ timelike).

1. $c = 0$: $u(s) = -\epsilon s^3/3$, $v(s) = 1/s$, $\kappa(s) = 2/s$, $s \neq 0$.



Spacelike (blue) and timelike (red) elastic curve with $\sigma = E = 0$

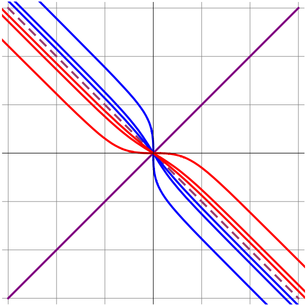
$\kappa(v) = 2v$

σ-elastica: $2\ddot{\kappa} - \kappa^3 - \sigma\kappa = 0$, $\sigma \in \mathbb{R}$.

Energy $E \in \mathbb{R}$: $E = \dot{\kappa}^2 - \frac{1}{4}\kappa^4 - \frac{\sigma}{2}\kappa^2$. $\boxed{E = \sigma^2/4\ ?}$

- $\mathcal{K}(v) = -\frac{\epsilon}{v^2+c}$, $c \in \mathbb{R}$ ($\epsilon = 1$ spacelike, $\epsilon = -1$ timelike).

2. $c > 0$: $u(s) = -\frac{\epsilon}{c}\left(\frac{s}{2} + \frac{\sin(2\sqrt{c}s)}{4\sqrt{c}}\right)$, $v(s) = -\sqrt{c}\tan(\sqrt{c}s)$.
   $\kappa(s) = -2\sqrt{c}\tan(\sqrt{c}s)$, $|s| < \pi/2\sqrt{c}$.



Spacelike (blue) and timelike (red) elastic curves with $\sigma = 4c > 0$ and $E = 4c^2$ ($c = 1, 2, 3$)

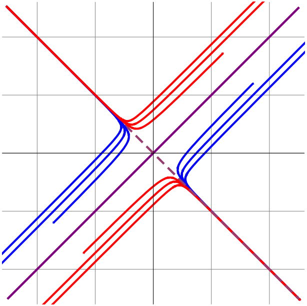
$\kappa(v) = 2v$

σ-elastica: $2\ddot{\kappa} - \kappa^3 - \sigma\kappa = 0$, $\sigma \in \mathbb{R}$.

Energy $E \in \mathbb{R}$: $E = \dot{\kappa}^2 - \frac{1}{4}\kappa^4 - \frac{\sigma}{2}\kappa^2$. $\boxed{E = \sigma^2/4\ ?}$

- $\mathcal{K}(v) = -\frac{\epsilon}{v^2+c}$, $c \in \mathbb{R}$ ($\epsilon = 1$ spacelike, $\epsilon = -1$ timelike).

3. $c < 0$: $u(s) = \frac{\epsilon}{c}\left(-\frac{s}{2} + \frac{\sinh(2\sqrt{-c}\,s)}{4\sqrt{-c}}\right)$, $v(s) = \sqrt{-c}\coth(\sqrt{-c}\,s)$.
   $\kappa(s) = 2\sqrt{-c}\coth(\sqrt{-c}s)$, $s \neq 0$.



Spacelike (blue) and timelike (red) elastic curves with $\sigma = 4c < 0$ and $E = 4c^2$ ($c = -1, -2, -3$)

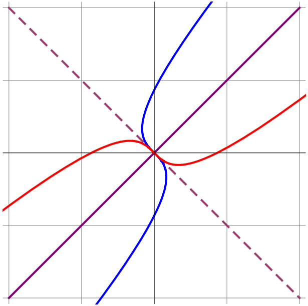
$\kappa(v) = 1/v^2$

$\kappa(v) = 1/v^2$

- $\mathcal{K}(v) = \epsilon v$ ($\epsilon = 1$ spacelike, $\epsilon = -1$ timelike)

$u(s) = 2\epsilon\sqrt{2s}\sqrt{s}/3, \quad v(s) = \sqrt{2s}, \quad \kappa(s) = \dfrac{1}{2s}, \; s > 0.$

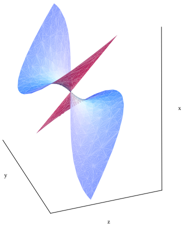
Graphs $u = \epsilon\, v^3/3$, $v > 0$ for $\epsilon = \pm 1$.



Spacelike (blue) and timelike (red) curve in $\mathbb{L}^2$
with $\mathcal{K}(v) = \epsilon v$, $\epsilon = \pm 1$.

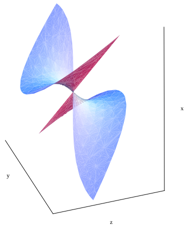
Generatrix of Enneper's surface of second kind

Kobayashi, 1993: Enneper's surface of second kind.
Rotation surface with lightlike axis $(1, 0, 1)$ and generatrix curve $x = \lambda(-t + t^3/3)$, $z = \lambda(t + t^3/3)$, $\lambda > 0$, at the xz-plane.



Generatrix of Enneper's surface of second kind

Kobayashi, 1993: Enneper's surface of second kind.
Rotation surface with lightlike axis $(1, 0, 1)$ and generatrix curve $x = \lambda(-t + t^3/3)$, $z = \lambda(t + t^3/3)$, $\lambda > 0$, at the xz-plane.



The generatrix curve of Enneper's surface for $\lambda = 1/2$ coincide with the graph $u = v^3/3$, $v > 0$.

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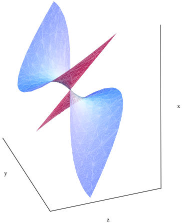

The generatrix curve of Enneper's surface for $\lambda = 1/2$ coincide with the graph $u = v^3/3$, $v > 0$.

The generatrix curve of the Enneper's surface of second kind, $u = v^3/3$, $v > 0$, is the only *spacelike* curve (up to dilations and u-translations) with geometric linear momentum $\mathcal{K}(v) = v$ (and curvature $\kappa(v) = 1/v^2$)

$\kappa(v) = 1/v^2$

- $\mathcal{K}(v) = \frac{-\epsilon v}{c\,v-1}$, $c \neq 0$ ($\epsilon = 1$ spacelike, $\epsilon = -1$ timelike)

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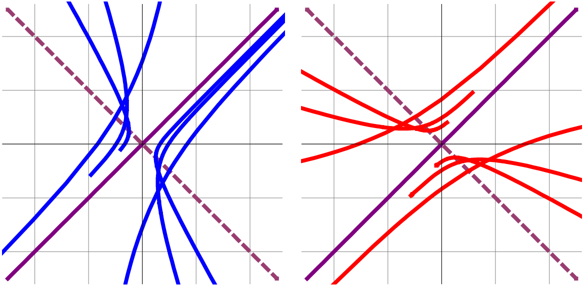
- $\mathcal{K}(v) = \frac{-\epsilon v}{c\,v-1}$, $c \neq 0$ ($\epsilon = 1$ spacelike, $\epsilon = -1$ timelike)

$$u = u(v) = \frac{\epsilon}{c^3}\left(c\,v - 1 - \frac{1}{c\,v-1} + 2\log(c\,v-1)\right),$$
$$v > 1/c \text{ if } c > 0,\ v < 1/c \text{ if } c < 0.$$



Spacelike curves with $\mathcal{K}(v) = -\frac{v}{c\,v-1}$ (left) and timelike curves with $\mathcal{K}(v) = \frac{v}{c\,v-1}$ (right).

$\kappa(v) = e^v$

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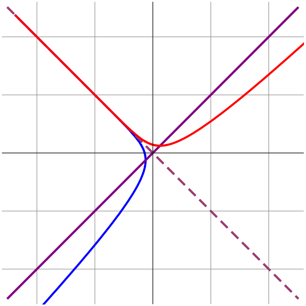
- $\mathcal{K}(v) = -\frac{\epsilon}{e^v + c}$, $c \in \mathbb{R}$ ($\epsilon = 1$ spacelike, $\epsilon = -1$ timelike)

$\kappa(v) = e^v$

- $\mathcal{K}(v) = -\frac{\epsilon}{e^v + c}$, $c \in \mathbb{R}$ ($\epsilon = 1$ spacelike, $\epsilon = -1$ timelike)
  1. $c = 0$: $u(s) = -\epsilon s^2/2$, $v(s) = -\log s$, $\kappa(s) = 1/s$, $s > 0$.

Graph $u = -\epsilon\, e^{-2v}/2$, $v \in \mathbb{R}$.

Translating-type soliton equation: $\kappa = g((1,1), N)$.



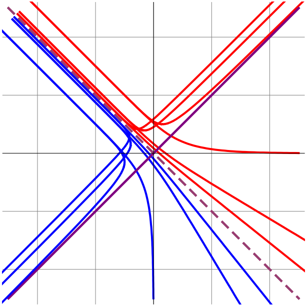
Spacelike (blue) and timelike (red)

Lorentzian grim-reapers, $\mathcal{K}(v) = -\frac{\epsilon}{e^v}$

$\kappa(v) = e^v$

- $\mathcal{K}(v) = -\frac{\epsilon}{e^v + c}$, $c \in \mathbb{R}$ ($\epsilon = 1$ spacelike, $\epsilon = -1$ timelike)

2. $c \neq 0$: $u(s) = -\frac{\epsilon}{c}\left(s + \frac{1}{c\, e^{cs}}\right)$, $v(s) = \log \frac{c}{e^{cs} - 1}$,
   $\kappa(s) = \frac{c}{e^{cs} - 1}$, $s > 0$



Spacelike curves (blue) and timelike curves (red)
with $\mathcal{K}(v) = -\frac{\epsilon}{e^v + c}$, $c \neq 0$

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References

I. Castro, I. Castro-Infantes and J. Castro-Infantes: *Curves in Lorentz-Minkowski plane: elasticae, catenaries and grim-reapers*. Open Math. **16** (2018), 747–776.

I. Castro, I. Castro-Infantes and J. Castro-Infantes: *Curves in Lorentz-Minkowski plane with curvature depending on their position*. Preprint 2018. arXiv:1806.09187 [math.DG].

References

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Thank you very much for your attention!