Jacobi Multipliers and Hamel's formalism[1]

Patrícia Santos

CMUC, University of Coimbra
ISEC, Coimbra Polytechnic

XXVII International Fall Workshop on Geometry and Physics
September 4, 2018

[1] Joint work with José F. Cariñena

Outline – Jacobi Multipliers and Hamel's formalism

§1 Motivation and geometric formalism

§2 Second-order differential equations and Lagrangians

§3 Jacobi multipliers in quasi-coordinates

§4 Jacobi Last Multiplier for a nonholonomic system

§1 Motivation and geometric formalism

The notion of Jacobi Multiplier (JM) was introduced by Jacobi in 1844 “as new broad principle in mechanics”, since then many authors developed and apply this mathematical concept (for an interesting historical account one can see the paper of L. C. Berrone & H. Giacomini[2]).

A JM is a function that plays an important role in the solution of systems of first-order ordinary differential equations. For systems of second-order differential equations,

$$\ddot{x}^i = f^i(x, \dot{x}), \tag{1}$$

the theory of JM can provide Lagrangian descriptions and it is particularly useful to find constants of motion (see the paper of J. F. Cariñena *et al.*[3] and references therein).

[2] *Inverse Jacobi multipliers*, Rend. Circ. Mat. Palermo **52** (2003) 77-130.

[3] *Jacobi multipliers, non-local symmetries and nonlinear oscillators*, J. Math. Phys. **56** (2015) 465206.

§1 Motivation and geometric formalism

The notion of Jacobi Multiplier (JM) was introduced by Jacobi in 1844 “as new broad principle in mechanics”, since then many authors developed and apply this mathematical concept (for an interesting historical account one can see the paper of L. C. Berrone & H. Giacomini[2]).

A JM is a function that plays an important role in the solution of systems of first-order ordinary differential equations. For systems of second-order differential equations,

$$\ddot{x}^i = f^i(x, \dot{x}), \tag{1}$$

the theory of JM can provide Lagrangian descriptions and it is particularly useful to find constants of motion (see the paper of J. F. Cariñena *et al.*[3] and references therein).

[2] *Inverse Jacobi multipliers*, Rend. Circ. Mat. Palermo **52** (2003) 77-130.

[3] *Jacobi multipliers, non-local symmetries and nonlinear oscillators*, J. Math. Phys. **56** (2015) 465206.

§1 Motivation and geometric formalism

Let (M, Ω) be an oriented n-dimensional manifold, where Ω stands for a volume form on M. Given a v.f. X on M we define the *divergence* of X as the unique function $\mathrm{div}(X) : M \to \mathbb{R}$ satisfying $\mathcal{L}_X \Omega = \mathrm{div}(X)\, \Omega$.

A *Jacobi Multiplier* (JM) for (X, Ω) is a non-vanishing function $\mu : M \to \mathbb{R}$ satisfying

$$\mathcal{L}_{\mu X} \Omega = 0, \tag{2}$$

that is, $\mathrm{div}(\mu X) = 0$, the v.f. μX is called *solenoidal*.

Since the volume form Ω is closed, Cartan's magic formula implies that

$$\mathcal{L}_{\mu X}(\Omega) = d(\mu\, i(X)\Omega). \tag{3}$$

Thus, μ is a JM for (X, Ω) iff the form $\mu\, i(X)\Omega$ is closed and iff $\mathcal{L}_X(\mu\, \Omega) = 0$.
As a consequence, given two volume forms on M such that $\Omega' = \eta\Omega$, if μ is a JM for (X, Ω) then $\mu' = \mu/\eta$ is a JM for (X, Ω').

§1 Motivation and geometric formalism

Let (M, Ω) be an oriented n-dimensional manifold, where Ω stands for a volume form on M. Given a v.f. X on M we define the *divergence* of X as the unique function $\mathrm{div}(X) : M \to \mathbb{R}$ satisfying $\mathcal{L}_X \Omega = \mathrm{div}(X)\,\Omega$.

A *Jacobi Multiplier* (JM) for (X, Ω) is a non-vanishing function $\mu : M \to \mathbb{R}$ satisfying

$$\mathcal{L}_{\mu X} \Omega = 0, \tag{2}$$

that is, $\mathrm{div}(\mu X) = 0$, the v.f. μX is called *solenoidal*.

Since the volume form Ω is closed, Cartan's magic formula implies that

$$\mathcal{L}_{\mu X}(\Omega) = d(\mu\, i(X)\Omega). \tag{3}$$

Thus, μ is a JM for (X, Ω) iff the form $\mu\, i(X)\Omega$ is closed and iff $\mathcal{L}_X(\mu\,\Omega) = 0$.
As a consequence, given two volume forms on M such that $\Omega' = \eta\Omega$, if μ is a JM for (X, Ω) then $\mu' = \mu/\eta$ is a JM for (X, Ω').

§1 Motivation and geometric formalism

Let (M, Ω) be an oriented n-dimensional manifold, where Ω stands for a volume form on M. Given a v.f. X on M we define the *divergence* of X as the unique function $\operatorname{div}(X) : M \to \mathbb{R}$ satisfying $\mathcal{L}_X \Omega = \operatorname{div}(X)\,\Omega$.

A *Jacobi Multiplier* (JM) for (X, Ω) is a non-vanishing function $\mu : M \to \mathbb{R}$ satisfying

$$\mathcal{L}_{\mu X} \Omega = 0, \tag{2}$$

that is, $\operatorname{div}(\mu X) = 0$, the v.f. μX is called *solenoidal*.

Since the volume form Ω is closed, Cartan's magic formula implies that

$$\mathcal{L}_{\mu X}(\Omega) = d(\mu\, i(X)\Omega). \tag{3}$$

Thus, μ is a JM for (X, Ω) iff the form $\mu\, i(X)\Omega$ is closed and iff $\mathcal{L}_X(\mu\,\Omega) = 0$.

As a consequence, given two volume forms on M such that $\Omega' = \eta\Omega$, if μ is a JM for (X, Ω) then $\mu' = \mu/\eta$ is a JM for (X, Ω').

§1 Motivation and geometric formalism

Let (M,Ω) be an oriented n-dimensional manifold, where Ω stands for a volume form on M. Given a v.f. X on M we define the *divergence* of X as the unique function $\mathrm{div}(X): M \to \mathbb{R}$ satisfying $\mathcal{L}_X\Omega = \mathrm{div}(X)\,\Omega$.

A *Jacobi Multiplier* (JM) for (X,Ω) is a non-vanishing function $\mu : M \to \mathbb{R}$ satisfying

$$\mathcal{L}_{\mu X}\Omega = 0, \tag{2}$$

that is, $\mathrm{div}(\mu X) = 0$, the v.f. μX is called *solenoidal*.

Since the volume form Ω is closed, Cartan's magic formula implies that

$$\mathcal{L}_{\mu X}(\Omega) = d(\mu\, i(X)\Omega). \tag{3}$$

Thus, μ is a JM for (X,Ω) iff the form $\mu\, i(X)\Omega$ is closed and iff $\mathcal{L}_X(\mu\,\Omega) = 0$.
As a consequence, given two volume forms on M such that $\Omega' = \eta\Omega$, if μ is a JM for (X,Ω) then $\mu' = \mu/\eta$ is a JM for (X,Ω').

§1 Motivation and geometric formalism

Furthermore, $\mathcal{L}_X$ is a derivation of degree zero then $\mathcal{L}_X(\mu\,\Omega) = (X\mu + \mu\,\mathrm{div}X)\Omega$, which proves equation (2) is equivalent to

$$X\mu + \mu\,\mathrm{div}(X) = 0 \tag{4}$$

Along the integral curves of X we obtain the generalised Liouville equation

$$\frac{d}{dt}\ln|\mu| + \mathrm{div}(X) = 0. \tag{5}$$

Jacobi multipliers are useful to find constants of motion... The knowledge of two JM μ_1 and μ_2 for X implies that $I = \mu_1/\mu_2$ is a constant of motion of X.

In the case of a 2-dimensional system of first-order differential equations, a Jacobi multiplier μ determines a first-integral of the system I (locally defined) by means of $\mu\, i(X)\Omega = dI$. Note that, a first-integral of a 2-dimensional system provides the general solution of the system.

§1 Motivation and geometric formalism

Furthermore, $\mathcal{L}_X$ is a derivation of degree zero then $\mathcal{L}_X(\mu\,\Omega) = (X\mu + \mu\,\mathrm{div}X)\Omega$, which proves equation (2) is equivalent to

$$X\mu + \mu\,\mathrm{div}(X) = 0 \tag{4}$$

Along the integral curves of X we obtain the generalised Liouville equation

$$\frac{d}{dt}\ln|\mu| + \mathrm{div}(X) = 0. \tag{5}$$

Jacobi multipliers are useful to find constants of motion... The knowledge of two JM μ_1 and μ_2 for X implies that $I = \mu_1/\mu_2$ is a constant of motion of X.

In the case of a 2-dimensional system of first-order differential equations, a Jacobi multiplier μ determines a first-integral of the system I (locally defined) by means of $\mu\, i(X)\Omega = dI$. Note that, a first-integral of a 2-dimensional system provides the general solution of the system.

§1 Motivation and geometric formalism

Furthermore, $\mathcal{L}_X$ is a derivation of degree zero then $\mathcal{L}_X(\mu\,\Omega) = (X\mu + \mu\,\mathrm{div}X)\Omega$, which proves equation (2) is equivalent to

$$X\mu + \mu\,\mathrm{div}(X) = 0 \tag{4}$$

Along the integral curves of X we obtain the generalised Liouville equation

$$\frac{d}{dt}\ln|\mu| + \mathrm{div}(X) = 0. \tag{5}$$

Jacobi multipliers are useful to find constants of motion... The knowledge of two JM μ_1 and μ_2 for X implies that $I = \mu_1/\mu_2$ is a constant of motion of X.

In the case of a 2-dimensional system of first-order differential equations, a Jacobi multiplier μ determines a first-integral of the system I (locally defined) by means of $\mu\, i(X)\Omega = dI$. Note that, a first-integral of a 2-dimensional system provides the general solution of the system.

§1 Motivation and geometic formalism

Example 1 - Liénard's equation and Chiellini's condition

Consider the classical Liénard equation[4]

$$\ddot{x} + \overbrace{f(x)\dot{x}}^{damping} + g(x) = 0. \tag{6}$$

This is a class of second-order differential equations (SODE) that can be used to model oscillating circuits, in which the damping term is proportional to the velocity and f is a non-vanishing function. Liénard's equation defines a v.f. given by

$$\Gamma = v\frac{\partial}{\partial x} - (f(x)v + g(x))\frac{\partial}{\partial v}. \tag{7}$$

Since $\mathrm{div}(\Gamma) = -f$, then μ is a JM of the system iff

$$\Gamma\mu - \mu f = 0. \tag{8}$$

[4] A. Liénard, *Étude des oscillations entretenues, Revue Générale de l'Électricité* **23** *(1928), 901-912 and 946-954.*

§1 Motivation and geometic formalism

Example 1 - Liénard's equation and Chiellini's condition

Consider the classical Liénard equation[4]

$$\ddot{x} + \overbrace{f(x)\dot{x}}^{damping} + g(x) = 0. \tag{6}$$

This is a class of second-order differential equations (SODE) that can be used to model oscillating circuits, in which the damping term is proportional to the velocity and f is a non-vanishing function. Liénard's equation defines a v.f. given by

$$\Gamma = v\frac{\partial}{\partial x} - (f(x)v + g(x))\frac{\partial}{\partial v}. \tag{7}$$

Since $\mathrm{div}(\Gamma) = -f$, then μ is a JM of the system iff

$$\Gamma\mu - \mu f = 0. \tag{8}$$

[4] A. Liénard, *Étude des oscillations entretenues, Revue Générale de l'Électricité* **23** *(1928), 901-912 and 946-954.*

§1 Motivation and geometric formalism

Example 1 - Liénard's equation and Chiellini's condition

Looking for a JM of the form $\mu = (v - W(x))^{1/s}$ we find out that μ is a JM iff

$$sW(x) = g(x)/f(x) \;\text{ and }\; W'(x) = -(1+s)f(x). \tag{9}$$

Combining the two equations we obtain a compatibility condition between f and g for the existence of a JM μ of the previously chosen form:

$$\frac{d}{dx}\left(\frac{g}{f}\right) = kf, \quad \text{with } k = -s(1+s). \tag{10}$$

The above condition is known as Chiellini's condition, and appear in this example as a consequence of the theory of Jacobi multipliers. Note that Chiellini's condition is used to integrate first kind Abel equations and SODE that can be reduced to them.[5]

[5] T. Harko, F. S. N. Lobo and M. K. Mak, *A Chiellini type integrability condition for the generalized first kind Abel differential equation*, Universal Journal of Applied Mathematics **1(2)** (2013) 101–104.

§2 Second-order differential equations and Lagrangians

Let (Q, g) be a n-dimensional Riemannian manifold, and let $\tau : TQ \to Q$ denote the tangent bundle of Q. Consider a coordinate system $(U, q^1, \ldots, q^n)$ on the open set U of Q. This coordinate system determines a basis $(\partial_{q^1}, \ldots, \partial_{q^n})$ of the tangent bundle at each point of $\tau^{-1}(U)$.

The choice of local coordinates on Q also defines an associated local coordinate system $(q^1, \ldots, q^n, v^1, \ldots, v^n)$ on the tangent bundle TQ, where we can consider a volume form locally given by

$$\Omega = dq^1 \wedge \cdots \wedge dq^n \wedge dv^1 \wedge \cdots \wedge dv^n. \tag{11}$$

The divergence of a vector field $X = h^i(q, v)\partial_{q^i} + f^i(q, v)\partial_{v^i}$ w.r.t. Ω turns out to be [6]

$$\operatorname{div}(X) = \frac{\partial h^i}{\partial q^i} + \frac{\partial f^i}{\partial v^i}. \tag{12}$$

[6] Einstein summation convention

§2 Second-order differential equations and Lagrangians

Let (Q, g) be a n-dimensional Riemannian manifold, and let $\tau : TQ \to Q$ denote the tangent bundle of Q. Consider a coordinate system $(U, q^1, \ldots, q^n)$ on the open set U of Q. This coordinate system determines a basis $(\partial_{q^1}, \ldots, \partial_{q^n})$ of the tangent bundle at each point of $\tau^{-1}(U)$.

The choice of local coordinates on Q also defines an associated local coordinate system $(q^1, \ldots, q^n, v^1, \ldots, v^n)$ on the tangent bundle TQ, where we can consider a volume form locally given by

$$\Omega = dq^1 \wedge \cdots \wedge dq^n \wedge dv^1 \wedge \cdots \wedge dv^n. \tag{11}$$

The divergence of a vector field $X = h^i(q,v)\partial_{q^i} + f^i(q,v)\partial_{v^i}$ w.r.t. Ω turns out to be [6]

$$\mathrm{div}(X) = \frac{\partial h^i}{\partial q^i} + \frac{\partial f^i}{\partial v^i}. \tag{12}$$

[6] Einstein summation convention

§2 Second-order differential equations and Lagrangians

Let (Q, g) be a n-dimensional Riemannian manifold, and let $\tau : TQ \to Q$ denote the tangent bundle of Q. Consider a coordinate system $(U, q^1, \ldots, q^n)$ on the open set U of Q. This coordinate system determines a basis $(\partial_{q^1}, \ldots, \partial_{q^n})$ of the tangent bundle at each point of $\tau^{-1}(U)$.

The choice of local coordinates on Q also defines an associated local coordinate system $(q^1, \ldots, q^n, v^1, \ldots, v^n)$ on the tangent bundle TQ, where we can consider a volume form locally given by

$$\Omega = dq^1 \wedge \cdots \wedge dq^n \wedge dv^1 \wedge \cdots \wedge dv^n. \tag{11}$$

The divergence of a vector field $X = h^i(q, v)\partial_{q^i} + f^i(q, v)\partial_{v^i}$ w.r.t. Ω turns out to be [6]

$$\mathrm{div}(X) = \frac{\partial h^i}{\partial q^i} + \frac{\partial f^i}{\partial v^i}. \tag{12}$$

[6] Einstein summation convention

§2 Second-order differential equations and Lagrangians

Suppose we have an autonomous SODE system $\ddot{q}^i = f^i(q, \dot{q})$. The system determines a v.f. Γ given in a neighbourhood of a point in TQ by

$$\Gamma = v^i \frac{\partial}{\partial q^i} + f^i(q, v) \frac{\partial}{\partial v^i}, \tag{13}$$

whose integral curves are the solutions of the first-order system

$$\begin{cases} \dot{q}^i = v^i \\ \dot{v}^i = f^i(q, v) \end{cases} . \tag{14}$$

Theorem

If the second-order system is determined by a regular Lagrangian $L \in C^\infty(TQ)$ then the determinant of the Lagrangian Hessian matrix in the velocities is a JM of (Γ, Ω).

§2 Second-order differential equations and Lagrangians

Suppose we have an autonomous SODE system $\ddot{q}^i = f^i(q, \dot{q})$. The system determines a v.f. Γ given in a neighbourhood of a point in TQ by

$$\Gamma = v^i \frac{\partial}{\partial q^i} + f^i(q, v) \frac{\partial}{\partial v^i}, \tag{13}$$

whose integral curves are the solutions of the first-order system

$$\begin{cases} \dot{q}^i = v^i \\ \dot{v}^i = f^i(q, v) \end{cases} . \tag{14}$$

Theorem

If the second-order system is determined by a regular Lagrangian $L \in C^\infty(TQ)$ then the determinant of the Lagrangian Hessian matrix in the velocities is a JM of (Γ, Ω).

§2 Second-order differential equations and Lagrangians

Particular cases:

- If the 1-dimensional system $\ddot{q} = f(q, \dot{q})$ admits a regular Lagrangian formulation, L, then we obtain the well known result that $\mu = \partial^2 L/\partial v^2$ is a JM and f is given by

$$f(q, v) = \frac{1}{\mu}\left(\frac{\partial L}{\partial q} - v\frac{\partial^2 L}{\partial q \partial v}\right). \tag{15}$$

  On the other hand, if a JM μ is known for a 1-dimensional SODE system, then there exists a regular Lagrangian for the system, unique defined up to addition of a gauge term, that verifies $\mu = \partial^2 L/\partial v^2$.

- For a mechanical system defined by a Lagrangian of the form $L = T - V$, where the kinetic energy $T(q, v) = \frac{1}{2} g_{ij}(q) v^i v^j$ is determined by the metric tensor g and $V(q)$ is the potential energy of the system, the determinant of the matrix with elements $g_{ij} = g(\partial_{q^i}, \partial_{q^j})$ is a JM for the SODE v.f.

§2 Second-order differential equations and Lagrangians

Particular cases:

- If the 1-dimensional system $\ddot{q} = f(q, \dot{q})$ admits a regular Lagrangian formulation, L, then we obtain the well known result that $\mu = \partial^2 L/\partial v^2$ is a JM and f is given by

$$f(q, v) = \frac{1}{\mu}\left(\frac{\partial L}{\partial q} - v\frac{\partial^2 L}{\partial q \partial v}\right). \tag{15}$$

  On the other hand, if a JM μ is known for a 1-dimensional SODE system, then there exists a regular Lagrangian for the system, unique defined up to addition of a gauge term, that verifies $\mu = \partial^2 L/\partial v^2$.

- For a mechanical system defined by a Lagrangian of the form $L = T - V$, where the kinetic energy $T(q, v) = \frac{1}{2} g_{ij}(q) v^i v^j$ is determined by the metric tensor g and $V(q)$ is the potential energy of the system, the determinant of the matrix with elements $g_{ij} = g(\partial_{q^i}, \partial_{q^j})$ is a JM for the SODE v.f.

§2 Second-order differential equations and Lagrangians

Proposition

Suppose that the generalized forces of a second-order system do not depend on the velocities. Then as the divergence of the dynamics v.f. Γ is zero, μ is JM for (Γ, Ω) iff μ is a constant of motion.

M. C. Nucci & P. G. L. Leach[7] proved for a n-dimensional SODE system that admits a Lagrangian formulation, if the generalized forces do not depend of the velocities then each function $\mu_{ij} = \partial^2 L/\partial v^i \partial v^j$ is a constant of motion and therefore a JM for Γ.

[7] *Jacobi's last multiplier and Lagrangians for multidimensional systems*, J. Math. Phys. **49** (2008) 073517 (8pp)

§3 Jacobi multipliers in quasi-coordinates

Choose a local basis $\{X_1, \ldots, X_n\}$ of vector fields on the (pseudo-)Riemmannian manifold (Q, g) and the dual basis $\{\alpha^1, \ldots, \alpha^n\}$. Any tangent vector $v \in T_q Q$ can be expressed uniquely as $v = w^j X_j(q)$. The real numbers $(w^1, \ldots, w^n)$ are called the *quasi-velocities* of v in the given basis and (q^i, w^i) is called the quasi-coordinates of v.

On the tangent bundle TQ we can consider a volume form locally given in quasi-coordinates as follows

$$\Omega = dq^1 \wedge \ldots \wedge dq^n \wedge dw^1 \wedge \ldots \wedge dw^n. \tag{16}$$

Associated to the quasi-coordinates we have a set of local functions on Q called Hamel's symbols given by

$$\gamma^k_{ml} = \beta^j_m \beta^i_l \left(\frac{\partial \alpha^k_j}{\partial x^i} - \frac{\partial \alpha^k_i}{\partial x^j} \right) \tag{17}$$

and determined by means of the commutations relations $[X_m, X_l] = \gamma^k_{ml} X_k$.

§3 Jacobi multipliers in quasi-coordinates

Choose a local basis $\{X_1, \ldots, X_n\}$ of vector fields on the (pseudo-)Riemmannian manifold (Q, g) and the dual basis $\{\alpha^1, \ldots, \alpha^n\}$. Any tangent vector $v \in T_qQ$ can be expressed uniquely as $v = w^j X_j(q)$. The real numbers $(w^1, \ldots, w^n)$ are called the *quasi-velocities* of v in the given basis and (q^i, w^i) is called the quasi-coordinates of v.

On the tangent bundle TQ we can consider a volume form locally given in quasi-coordinates as follows

$$\Omega = dq^1 \wedge \ldots \wedge dq^n \wedge dw^1 \wedge \ldots \wedge dw^n. \tag{16}$$

Associated to the quasi-coordinates we have a set of local functions on Q called Hamel's symbols given by

$$\gamma^k_{ml} = \beta^j_m \beta^i_l \left(\frac{\partial \alpha^k_j}{\partial x^i} - \frac{\partial \alpha^k_i}{\partial x^j} \right) \tag{17}$$

and determined by means of the commutations relations $[X_m, X_l] = \gamma^k_{ml} X_k$.

§3 Jacobi multipliers in quasi-coordinates

Choose a local basis $\{X_1, \ldots, X_n\}$ of vector fields on the (pseudo-)Riemmannian manifold (Q, g) and the dual basis $\{\alpha^1, \ldots, \alpha^n\}$. Any tangent vector $v \in T_qQ$ can be expressed uniquely as $v = w^j X_j(q)$. The real numbers $(w^1, \ldots, w^n)$ are called the *quasi-velocities* of v in the given basis and (q^i, w^i) is called the quasi-coordinates of v.

On the tangent bundle TQ we can consider a volume form locally given in quasi-coordinates as follows

$$\Omega = dq^1 \wedge \ldots \wedge dq^n \wedge dw^1 \wedge \ldots \wedge dw^n. \tag{16}$$

Associated to the quasi-coordinates we have a set of local functions on Q called Hamel's symbols given by

$$\gamma^k_{ml} = \beta^j_m \beta^i_l \left(\frac{\partial \alpha^k_j}{\partial x^i} - \frac{\partial \alpha^k_i}{\partial x^j}\right) \tag{17}$$

and determined by means of the commutations relations $[X_m, X_l] = \gamma^k_{ml} X_k$.

§3 Jacobi multipliers in quasi-coordinates

In quasi-coordinates the equations of motion of a conservative SODE system, determined by a regular Lagrangian $\mathscr{L}(q,w) = L(q, v(q,w))$, are the so called Boltzmann-Hamel equations:

$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial w^l}\right) - \beta_l^k \frac{\partial \mathscr{L}}{\partial q^k} - w^m \gamma_{ml}^k \frac{\partial \mathscr{L}}{\partial w^k} = 0, \quad l = 1, \ldots, n, \tag{18}$$

The Boltzmann-Hamel equations determine the v.f. $\Gamma = \dot{q}^i \frac{\partial}{\partial q^i} + \dot{w}^i \frac{\partial}{\partial w^i}$, with

$$\begin{cases} \dot{q}^i = w^j \beta_j^i \\ \dot{w}^i = \mathscr{W}^{il}\left(\beta_l^k \frac{\partial \mathscr{L}}{\partial q^k} + w^m \gamma_{ml}^k \frac{\partial \mathscr{L}}{\partial w^k} - w^m \beta_m^k \frac{\partial^2 \mathscr{L}}{\partial q^k \partial w^l}\right) \end{cases}, \tag{19}$$

where $\mathscr{W}^{ik}$ represents the inverse matrix of the Lagrangian Hessian matrix in quasi-velocities $\mathscr{W} = \left[\frac{\partial^2 \mathscr{L}}{\partial w^i \partial w^j}\right]$.

§3 Jacobi multipliers in quasi-coordinates

In quasi-coordinates the equations of motion of a conservative SODE system, determined by a regular Lagrangian $\mathscr{L}(q,w) = L(q, v(q,w))$, are the so called Boltzmann-Hamel equations:

$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial w^l}\right) - \beta^k_l \frac{\partial \mathscr{L}}{\partial q^k} - w^m \gamma^k_{ml} \frac{\partial \mathscr{L}}{\partial w^k} = 0, \quad l = 1, \ldots, n, \tag{18}$$

The Boltzmann-Hamel equations determine the v.f. $\Gamma = \dot{q}^i \frac{\partial}{\partial q^i} + \dot{w}^i \frac{\partial}{\partial w^i}$, with

$$\begin{cases} \dot{q}^i = w^j \beta^i_j \\ \dot{w}^i = \mathscr{W}^{il}\left(\beta^k_l \frac{\partial \mathscr{L}}{\partial q^k} + w^m \gamma^k_{ml} \frac{\partial \mathscr{L}}{\partial w^k} - w^m \beta^k_m \frac{\partial^2 \mathscr{L}}{\partial q^k \partial w^l}\right) \end{cases} , \tag{19}$$

where $\mathscr{W}^{ik}$ represents the inverse matrix of the Lagrangian Hessian matrix in quasi-velocities $\mathscr{W} = \left[\frac{\partial^2 \mathscr{L}}{\partial w^i \partial w^j}\right]$.

§3 Jacobi multipliers in quasi-coordinates

Theorem

If the second-order conservative system is determined by a regular Lagrangian $\mathscr{L}(q,w)$, then the determinant of the product $\alpha\mathscr{W}$ is a Jacobi multiplier for (Γ,Ω).

The above result is the Lagrangian equivalent of the one proved by Q. K. Ghori[8] for the Hamiltonian formalism in quasi-coordinates. Ghori proved that the determinant of α is a Jacobi multiplier for the Hamel's equations.

[8] *Jacobi's multiplier for Poincaré's equations*, Acta Mechanic Sinica, **10** nº 1 (1994), 70–72.

§3 Jacobi multipliers in quasi-coordinates

Example 2 - A JM for the Kepler problem in quasi-coordinates

Consider a particle P of mass $m = 1$ moving in a plane under the action of a central force $F(r) = -\gamma m m'/r^2$ on the direction of a fixed point O of mass $m' \gg m$, where $\gamma > 0$ and $r = \mathrm{dist}(O, P)$. The configuration space of the system is $Q = \mathbb{R}^2 - \{O\}$.

Let θ be the angle that the line OP makes with a fixed direction on the plane and consider the quasi-velocities: $w_r = \dot{r}$ and $w_\theta = r^2\dot{\theta}$. The dynamics in quasi-coordinates is determined by the Lagrangian

$$\mathscr{L}(r, \theta, w_r, w_\theta) = \frac{1}{2}\left(w_r^2 + \frac{w_\theta^2}{r^2}\right) + \frac{\gamma m'}{r}. \tag{20}$$

Using the previous theorem we have that $\mu = \det(\alpha\mathscr{W}) = 1$ is a Jacobi multiplier for the second-order system in quasi-coordinates. In fact the dynamics v.f.

$$\Gamma = w_r \frac{\partial}{\partial r} + \frac{w_\theta}{r^2}\frac{\partial}{\partial \theta} + \left(-\frac{\gamma m'}{r^2} + \frac{w_\theta^2}{r^3}\right)\partial_{w_r} \tag{21}$$

is a divergence-free v.f., so a JM is a constant of motion.

§3 Jacobi multipliers in quasi-coordinates
Example 2 - A JM for the Kepler problem in quasi-coordinates

Consider a particle P of mass $m = 1$ moving in a plane under the action of a central force $F(r) = -\gamma mm'/r^2$ on the direction of a fixed point O of mass $m' \gg m$, where $\gamma > 0$ and $r = \text{dist}(O, P)$. The configuration space of the system is $Q = \mathbb{R}^2 - \{O\}$.

Let θ be the angle that the line OP makes with a fixed direction on the plane and consider the quasi-velocities: $w_r = \dot{r}$ and $w_\theta = r^2\dot{\theta}$. The dynamics in quasi-coordinates is determined by the Lagrangian

$$\mathscr{L}(r, \theta, w_r, w_\theta) = \frac{1}{2}\left(w_r^2 + \frac{w_\theta^2}{r^2}\right) + \frac{\gamma m'}{r}. \tag{20}$$

Using the previous theorem we have that $\mu = \det(\alpha\mathscr{W}) = 1$ is a Jacobi multiplier for the second-order system in quasi-coordinates. In fact the dynamics v.f.

$$\Gamma = w_r\frac{\partial}{\partial r} + \frac{w_\theta}{r^2}\frac{\partial}{\partial\theta} + \left(-\frac{\gamma m'}{r^2} + \frac{w_\theta^2}{r^3}\right)\partial_{w_r} \tag{21}$$

is a divergence-free v.f., so a JM is a constant of motion.

§3 Jacobi multipliers in quasi-coordinates

Example 2 - A JM for the Kepler problem in quasi-coordinates

Consider a particle P of mass $m = 1$ moving in a plane under the action of a central force $F(r) = -\gamma m m'/r^2$ on the direction of a fixed point O of mass $m' \gg m$, where $\gamma > 0$ and $r = \mathrm{dist}(O, P)$. The configuration space of the system is $Q = \mathbb{R}^2 - \{O\}$.

Let θ be the angle that the line OP makes with a fixed direction on the plane and consider the quasi-velocities: $w_r = \dot{r}$ and $w_\theta = r^2\dot{\theta}$. The dynamics in quasi-coordinates is determined by the Lagrangian

$$\mathscr{L}(r, \theta, w_r, w_\theta) = \frac{1}{2}\left(w_r^2 + \frac{w_\theta^2}{r^2}\right) + \frac{\gamma m'}{r}. \tag{20}$$

Using the previous theorem we have that $\mu = \det(\alpha\mathscr{W}) = 1$ is a Jacobi multiplier for the second-order system in quasi-coordinates. In fact the dynamics v.f.

$$\Gamma = w_r \frac{\partial}{\partial r} + \frac{w_\theta}{r^2}\frac{\partial}{\partial \theta} + \left(-\frac{\gamma m'}{r^2} + \frac{w_\theta^2}{r^3}\right)\partial_{w_r} \tag{21}$$

is a divergence-free v.f., so a JM is a constant of motion.

§4 Jacobi Last Multiplier for a nonholonomic system

Consider a regular nonholonomic system with a set of linear constraints, which defines a rank r vector subbundle $\mathscr{D}$ of TQ called the *constraint submanifold*.

In quasi-coordinates $(x^i, w^j) = (x^i, w^a, w^A)$ on TQ the equations that defines the constraint manifold $\mathscr{D}$ are simply $w^A = 0$, with $A = n - r + 1, \ldots, n$, and (x^i, w^a) are the coordinates for $\mathscr{D}$. The annihilator $\mathscr{D}$ is generated by the set of 1-forms $\{\alpha^A \mid A = n' + 1, \ldots, n\}$, with $n' = n - r$.

The evolution of the nonholonomic system is given by the integral curves of a vector field Γ tangent to $\mathscr{D}$ that satisfies the Lagrange–d'Alembert equation:

$$i_\Gamma \omega_L - dE_L = -\lambda_A \, \tau^* \alpha^A, \tag{22}$$

where $\lambda_A \in C^\infty(TM)$ are the Lagrangian multipliers of the system, determined by the *tangency conditions* $\mathcal{L}_\Gamma w^A = 0$, for all $A = n' + 1, \ldots, n$.

§4 Jacobi Last Multiplier for a nonholonomic system

Consider a regular nonholonomic system with a set of linear constraints, which defines a rank r vector subbundle $\mathscr{D}$ of TQ called the *constraint submanifold*.

In quasi-coordinates $(x^i, w^j) = (x^i, w^a, w^A)$ on TQ the equations that defines the constraint manifold $\mathscr{D}$ are simply $w^A = 0$, with $A = n - r + 1, \ldots, n$, and (x^i, w^a) are the coordinates for $\mathscr{D}$. The annihilator $\mathscr{D}$ is generated by the set of 1-forms $\{\alpha^A \mid A = n' + 1, \ldots, n\}$, with $n' = n - r$.

The evolution of the nonholonomic system is given by the integral curves of a vector field Γ tangent to $\mathscr{D}$ that satisfies the Lagrange–d'Alembert equation:

$$i_\Gamma \omega_L - dE_L = -\lambda_A \, \tau^* \alpha^A, \tag{22}$$

where $\lambda_A \in C^\infty(TM)$ are the Lagrangian multipliers of the system, determined by the *tangency conditions* $\mathcal{L}_\Gamma w^A = 0$, for all $A = n' + 1, \ldots, n$.

§4 Jacobi Last Multiplier for a nonholonomic system

The integral curves of the solution $\Gamma|_{\mathscr{D}} = w^a X_a + \dot{w}^a \frac{\partial}{\partial w^a}$ satisfies the system

$$\begin{cases} \dot{q}^i = \beta^i_a w^a \\ \dot{w}^a = \mathscr{W}^{ab} \left(\beta^k_b \dfrac{\partial \mathscr{L}}{\partial q^k} + w^c \gamma^k_{cb} \dfrac{\partial \mathscr{L}}{\partial w^k} - w^c \beta^k_c \dfrac{\partial^2 \mathscr{L}}{\partial q^k \partial w^b} \right) \\ w^A = 0 \end{cases}, \tag{23}$$

The divergence of $\Gamma|_{\mathscr{D}}$ w.r.t. the volume form Ω is given by

$$\operatorname{div}(\Gamma|_{\mathscr{D}}) = w^a \frac{\partial \beta^i_a}{\partial q^i} + \frac{\partial \dot{w}^a}{\partial w^a}. \tag{24}$$

Notice that

$$\mathcal{L}_\Gamma \omega_{\mathscr{L}} = -d(\lambda_A \tau^* \alpha^A) \neq 0.$$

In general, $\mathcal{L}_\Gamma \Omega' \neq 0$, with $\Omega' = \omega_{\mathscr{L}}^{\wedge n}$, and this means that the divergence of the solution Γ w.r.t. to Ω' is not zero.

§4 Jacobi Last Multiplier for a nonholonomic system

The integral curves of the solution $\varGamma|_{\mathscr{D}} = w^a X_a + \dot{w}^a \frac{\partial}{\partial w^a}$ satisfies the system

$$\begin{cases} \dot{q}^i = \beta^i_a w^a \\ \dot{w}^a = \mathscr{W}^{ab}\left(\beta^k_b \dfrac{\partial \mathscr{L}}{\partial q^k} + w^c \gamma^k_{cb} \dfrac{\partial \mathscr{L}}{\partial w^k} - w^c \beta^k_c \dfrac{\partial^2 \mathscr{L}}{\partial q^k \partial w^b}\right) \\ w^A = 0 \end{cases}, \tag{23}$$

The divergence of $\varGamma|_{\mathscr{D}}$ w.r.t. the volume form $\varOmega$ is given by

$$\operatorname{div}(\varGamma|_{\mathscr{D}}) = w^a \frac{\partial \beta^i_a}{\partial q^i} + \frac{\partial \dot{w}^a}{\partial w^a}. \tag{24}$$

Notice that

$$\mathcal{L}_{\varGamma} \omega_{\mathscr{L}} = -d(\lambda_A \tau^* \alpha^A) \neq 0.$$

In general, $\mathcal{L}_{\varGamma} \varOmega' \neq 0$, with $\varOmega' = \omega_{\mathscr{L}}^{\wedge n}$, and this means that the divergence of the solution $\varGamma$ w.r.t. to Ω' is not zero.

§4 Jacobi Last Multiplier for a nonholonomic system

Example 3 - Motion of a free particle on $\mathbb{R}^3$ with a constraint

Consider the motion of a free particle of unitary mass in the configuration space $\mathbb{R}^3$, with the linear constraint $w_z = v_z - yv_x$. Let (x, y, z, w_x, w_y, w_z) be a system of quasi-coordinates on $T\mathbb{R}^3$, where: $w_x = v_x,\ w_y = v_y,\ w_z = v_z - yv_x$.

The motion of the free particle is defined by the regular Lagrangian function

$$\mathscr{L}(x, y, z, w_x, w_y, w_z) = \frac{1}{2}(w_x^2 + w_y^2 + (w_z + yw_x)^2). \tag{25}$$

The transformation matrix α and the Lagrangian Hessian matrix in quasi-velocities $\mathscr{W}$ are given by

$$\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -y & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathscr{W} = \begin{pmatrix} 1+y^2 & 0 & y \\ 0 & 1 & 0 \\ y & 0 & 1 \end{pmatrix}. \tag{26}$$

Then, $\det(\alpha\mathscr{W}) = 1$ is a Jacobi multiplier for the free system whose solution is given by the v.f. $\Gamma_{\mathscr{L}} = w_x\,\partial_x + w_y\,\partial_y + (w_z + yw_x)\,\partial_z - w_xw_y\,\partial_{w_z}$.

§4 Jacobi Last Multiplier for a nonholonomic system
Example 3 - Motion of a free particle on $\mathbb{R}^3$ with a constraint

Consider the motion of a free particle of unitary mass in the configuration space $\mathbb{R}^3$, with the linear constraint $w_z = v_z - y v_x$. Let (x, y, z, w_x, w_y, w_z) be a system of quasi-coordinates on $T\mathbb{R}^3$, where: $w_x = v_x,\ w_y = v_y,\ w_z = v_z - y v_x$.

The motion of the free particle is defined by the regular Lagrangian function

$$\mathscr{L}(x, y, z, w_x, w_y, w_z) = \frac{1}{2}\left(w_x^2 + w_y^2 + (w_z + y w_x)^2\right). \tag{25}$$

The transformation matrix α and the Lagrangian Hessian matrix in quasi-velocities $\mathscr{W}$ are given by

$$\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -y & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathscr{W} = \begin{pmatrix} 1+y^2 & 0 & y \\ 0 & 1 & 0 \\ y & 0 & 1 \end{pmatrix}. \tag{26}$$

Then, $\det(\alpha \mathscr{W}) = 1$ is a Jacobi multiplier for the free system whose solution is given by the v.f. $\Gamma_{\mathscr{L}} = w_x\,\partial_x + w_y\,\partial_y + (w_z + y w_x)\,\partial_z - w_x w_y\,\partial_{w_z}$.

§4 Jacobi Last Multiplier for a nonholonomic system
Example 3 - Motion of a free particle on $\mathbb{R}^3$ with a constraint

The nonholonomic system is given by the v.f. $\Gamma = \Gamma_{\mathscr{L}} + \lambda Z$ on $\mathscr{D}$, where λ is the Lagrange multiplier associated to the constraint w_z and Z is a vertical vector field determined by the equation $i_Z\omega_{\mathscr{L}} = -\tau^*\alpha^3$, explicitly:

$$\lambda = \frac{w_x w_y}{1+y^2}\,;$$

$$Z = -y\,\frac{\partial}{\partial w_x} + (1+y^2)\,\frac{\partial}{\partial w_z}\,;$$

$$\Gamma|_{\mathscr{D}} = w_x\,\frac{\partial}{\partial x} + w_y\,\frac{\partial}{\partial y} + y w_x\,\frac{\partial}{\partial z} - \frac{y\,w_x w_y}{1+y^2}\,\frac{\partial}{\partial w_x}.$$

The divergence of the solution $\Gamma|_{\mathscr{D}}$ w.r.t. the volume form Ω is non-zero:

$$\mathrm{div}(\Gamma|_{\mathscr{D}}) = -\frac{y w_y}{1+y^2}. \tag{27}$$

§4 Jacobi Last Multiplier for a nonholonomic system

Example 3 - Motion of a free particle on $\mathbb{R}^3$ with a constraint

The nonholonomic system is given by the v.f. $\mathit{\Gamma} = \mathit{\Gamma}_{\mathscr{L}} + \lambda Z$ on $\mathscr{D}$, where λ is the Lagrange multiplier associated to the constraint w_z and Z is a vertical vector field determined by the equation $i_Z \omega_{\mathscr{L}} = -\tau^* \alpha^3$, explicitly:

$$\lambda = \frac{w_x w_y}{1+y^2} \, ;$$

$$Z = -y \frac{\partial}{\partial w_x} + (1+y^2) \frac{\partial}{\partial w_z} \, ;$$

$$\Gamma|_{\mathscr{D}} = w_x \frac{\partial}{\partial x} + w_y \frac{\partial}{\partial y} + y w_x \frac{\partial}{\partial z} - \frac{y \, w_x w_y}{1+y^2} \frac{\partial}{\partial w_x} .$$

The divergence of the solution $\Gamma|_{\mathscr{D}}$ w.r.t. the volume form $\mathit{\Omega}$ is non-zero:

$$\operatorname{div}(\mathit{\Gamma}|_{\mathscr{D}}) = -\frac{y w_y}{1+y^2} . \qquad (27)$$

§4 Jacobi Last Multiplier for a nonholonomic system

Example 3 - Motion of a free particle on $\mathbb{R}^3$ with a constraint

However, the equation (27) implies that

$$\frac{d}{dt}\ln(\sqrt{1+y^2}) + \mathrm{div}(\Gamma|_{\mathscr{D}}) = 0. \tag{28}$$

So $\mu = \sqrt{1+y^2}$ is a Jacobi multiplier for $(\Gamma|_{\mathscr{D}}, \Omega)$.

Additionally, we can prove that $\eta = w_x^{-1}$ is also a Jacobi multiplier for $(\Gamma|_{\mathscr{D}}, \Omega)$, then $I = \mu/\eta = \sqrt{1+y^2}\, w_x$ is a constant of motion of the nonholonomic system.

§4 Jacobi Last Multiplier for a nonholonomic system

Theorem (JLM)

Suppose that the nonholonomic system has $s = 2n - r - 2 = n + n' - 2$ *linear independent constants of motion* $I_1, I_2, \ldots, I_s$*, and let* J *be the determinant of the Jacobian matrix of the function* $I = (I_1, \ldots, I_s)$ *w.r.t. the non-free coordinates. If* μ *is a Jacobi multiplier for* (Γ, Ω)*, then,* $\widehat{\mu} = \mu/J$ *is the Jacobi last multiplier for the reduced 2-dimensional system in the free coordinates,*

Remark as a corollary that we can apply the result even if the system has zero constraints, i.e. $r = 0$ and $n' = n - r = n$; in this case, the quasi-velocities are true velocities. The theorem can be applied for all system of second -order differential equations and it is a generalization of Jacobi's result (see e.g. the paper of L.R. Berrone & H. Giacomini[9], p. 81).

[9] *Inverse Jacobi multipliers*, Rend. Circ. Mat. Palermo **52** (2003) 77-130.

§4 Jacobi Last Multiplier for a nonholonomic system

Theorem (JLM)

Suppose that the nonholonomic system has $s = 2n - r - 2 = n + n' - 2$ *linear independent constants of motion* $I_1, I_2, \ldots, I_s$, *and let* J *be the determinant of the Jacobian matrix of the function* $I = (I_1, \ldots, I_s)$ *w.r.t. the non-free coordinates. If* μ *is a Jacobi multiplier for* (Γ, Ω), *then,* $\widehat{\mu} = \mu / J$ *is the Jacobi last multiplier for the reduced 2-dimensional system in the free coordinates,*

Remark as a corollary that we can apply the result even if the system has zero constraints, i.e. $r = 0$ and $n' = n - r = n$; in this case, the quasi-velocities are true velocities. The theorem can be applied for all system of second -order differential equations and it is a generalization of Jacobi's result (see e.g. the paper of L.R. Berrone & H. Giacomini[9], p. 81).

[9] *Inverse Jacobi multipliers*, Rend. Circ. Mat. Palermo **52** (2003) 77-130.

§4 Jacobi Last Multiplier for a nonholonomic system
Example 4 - JLM for the Kepler problem

Recal that the SODE system is determined by the v.f.

$$\Gamma = w_r \frac{\partial}{\partial r} + \frac{w_\theta}{r^2}\frac{\partial}{\partial \theta} + \left(-\frac{\gamma m'}{r^2} + \frac{w_\theta^2}{r^3}\right)\partial_{w_r} \tag{29}$$

and $\mu = 1$ is a constant Jacobi multiplier for (Γ, Ω). The system have two obvious constants of motion, the energy of the system $I_1 = E_L$ and the area swept by unity of time by the line OP given by $I_2 = w_\theta$.

Taken (r, θ) as the free coordinates, we obtain the following Jacobian $J = w_r$, and then $\widehat{\mu} = 1/w_r$ is the Jacobi last multiplier of the reduced system, where w_r is consider as an function in the free coordinates.

§4 Jacobi Last Multiplier for a nonholonomic system
Example 4 - JLM for the Kepler problem

Recal that the SODE system is determined by the v.f.

$$\Gamma = w_r \frac{\partial}{\partial r} + \frac{w_\theta}{r^2} \frac{\partial}{\partial \theta} + \left(-\frac{\gamma m'}{r^2} + \frac{w_\theta^2}{r^3} \right) \partial_{w_r} \tag{29}$$

and $\mu = 1$ is a constant Jacobi multiplier for (Γ, Ω). The system have two obvious constants of motion, the energy of the system $I_1 = E_L$ and the area swept by unity of time by the line OP given by $I_2 = w_\theta$.

Taken (r, θ) as the free coordinates, we obtain the following Jacobian $J = w_r$, and then $\widehat{\mu} = 1/w_r$ is the Jacobi last multiplier of the reduced system, where w_r is consider as an function in the free coordinates.