Quantum Control on the Boundary

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Outline

- Controllability of Quantum Systems
- Quantum Control on the boundary

- Time dependent Schrödinger Equation

$$i\frac{\partial \Psi}{\partial t} = H(t)\Psi$$

- $H(t)$ is a family of self-adjoint operators
- The solution of the equation is given in terms of a unitary propagator
  - $U : \mathbb{R} \times \mathbb{R} \to \mathcal{U}(\mathcal{H})$
  - $U(t,t) = \mathbb{I}_{\mathcal{H}}$
  - $U(t,s)U(s,r) = U(t,r)$
- $\Psi(t) = U(t,t_0)\Psi_0$ is a solution of Schrödinger's Equation with initial value Ψ_0

- Finite dimensional quantum System $\mathcal{H} = \mathbb{C}^n$
- Simple situation. Linear controls:

$$i\frac{\partial \Psi}{\partial t} = \left(H_0 + c(t)H_1\right)\Psi$$

- H_0, H_1 self-adjoint operators (Hermitean matrices).
- $c : \mathbb{R} \to \mathcal{C}$ Space of controls
- Use the controls to steer the state of the system from $\Psi_0 \to \Psi_f$.

Controllability of finite dimensional Quantum systems

- Study the dynamical Lie algebra:

$$\mathfrak{Lie}\{iH_0, iH_1\}$$

- The reachable set of Ψ_0 is the orbit through Ψ_0 of the exponential map of the dynamical Lie algebra.

- The finite dimensional quantum system is controllable if the dynamical Lie algebra is the Lie algebra of $U(N)$.

Harmonic Oscillator

$$i\frac{\partial\Psi}{\partial t} = -\frac{1}{2}\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2} + \frac{1}{2}x^2\Psi + c(t)x\Psi = \left[\frac{1}{2}(p^2+q^2) + c(t)q\right]\Psi$$

$$p\Psi = -i\frac{\mathrm{d}\Psi}{\mathrm{d}x} \qquad q\Psi = x\Psi(x)$$

- Harmonic Oscillator algebra:

$$a^\dagger = \frac{1}{\sqrt{2}}(q - ip) \qquad a = \frac{1}{\sqrt{2}}(q + ip) \qquad N = a^\dagger a$$

$$N|n\rangle = n|n\rangle \qquad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \qquad a|n\rangle = \sqrt{n}|n-1\rangle$$

Example: Truncation of the Harmonic Oscillator

H_0 Harmonic Oscillator

H_1

$$i\frac{\partial\Psi}{\partial t} = \boxed{-\frac{1}{2}\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2} + \frac{1}{2}x^2\Psi} + c(t)\boxed{x\Psi} = \left[\frac{1}{2}(p^2+q^2) + c(t)q\right]\Psi$$

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- Generators of the dynamic:

$$H_0 = N + \frac{1}{2} \qquad H_1 = \frac{1}{\sqrt{2}}(a^\dagger + a)$$

- Finite-dimensional approximation by the first n eigen-states

$$(H_0^n)_{ij} = \langle i|H_0|j\rangle \qquad (H_1^n)_{ij} = \langle i|H_0|j\rangle$$

Example: Truncation of the Harmonic Oscillator

- Finite-dimensional approximation by the first n eigen-states

$$(H_0^n)_{ij} = \langle i|H_0|j\rangle \qquad (H_1^n)_{ij} = \langle i|H_0|j\rangle$$

- The finite dimensional approximation is controllable for all n

$$\dim \mathfrak{Lie}\{iH_0, iH_1\} = n^2$$

- Generators of the dynamic:

$$H_0 = N + \frac{1}{2} \qquad H_1 = \frac{1}{\sqrt{2}}(a^\dagger + a)$$

- Dynamical Lie Algebra of the Harmonic Oscillator

$$[a, a^\dagger] = \mathbb{I} \qquad [N, a] = -a \qquad [N, a^\dagger] = a^\dagger$$

$$[iH_0, iH_1] = -\frac{1}{\sqrt{2}}[N, a^\dagger + a] = -\frac{1}{\sqrt{2}}(a^\dagger - a) = ip = iH_2$$

Controllability of the Harmonic Oscillator

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$$[iH_0, iH_1] = iH_2 \qquad [iH_0, iH_2] = iH_1 \qquad [iH_1, iH_2] = i\mathbb{I} = iH_3$$

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- Four dimensional Lie algebra!
- The infinite dimensional Harmonic Oscillator is not controllable.

- Why is it controllable for finite dimensions?
- Consider the 3-level truncation ($n = 0, 1, 2$)

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

$$a|n\rangle = \sqrt{n}|n-1\rangle$$

$$a^\dagger = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix}$$

$$a = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{bmatrix}$$

- What happens with the dynamical Lie algebra?

$$q = \frac{1}{\sqrt{2}}(a^\dagger + a)$$

$$p = \frac{i}{\sqrt{2}}(a^\dagger - a)$$

$$[q, p] = i \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Approximate Controllability: A linear control system is approximately controllable if for every Ψ_0, $\Psi_1 \in \mathcal{S}$ and every $\epsilon > 0$ there exist $T > 0$ and $c(t) \subset \mathcal{C}$ such that

$$\|\Psi_1 - U(T, t_0)\Psi_0\| < \epsilon$$

- Reasonable for infinite dimensions
- Hilbert Space is defined as equivalence classes of convergent sequences
- Is natural to expect this if one has exact controllability of every finite dimensional subsystem

Approximate Controllability

- Consider the Linear Control System:

$$i\frac{\partial\Psi}{\partial t} = \left(H_0 + c(t)H_1\right)\Psi$$

  - H_0, H_1 are self-adjoint.
  - $\{\Phi_n\}_{n\in\mathbb{N}}$ O.N.B of eigenvectors of H_0
  - $\Phi_n \in \mathcal{D}(H_1)$ for every $n \in \mathbb{N}$

- The linear control system is approximately controllable with piecewise constant controls if [Chambrion, Mason, Sigalotti, Boscain 2009]:
  - $(\lambda_{n+1} - \lambda_n)_{n\in\mathbb{N}}$ are $\mathbb{Q}$-linearly independent.
  - $\langle H_1\Phi_n, \Phi_{n+1}\rangle \neq 0$ for any $n \in \mathbb{N}$

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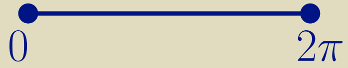
- $H(t)$ is a family of different self-adjoint extension of the same operator

$$\left(H, \mathcal{D}\left(c(t)\right)\right)$$

- Advantage: There is no need to apply an external field
- Problem: Even the existence of solutions of the dynamics is compromised.



$$H_0 = -\frac{\mathsf{d}^2}{\mathsf{d}x^2}$$

$$\mathcal{D}_\alpha = \left\{ \phi \in \mathcal{L}^2 \middle| \left\|\frac{\mathsf{d}^2\phi}{\mathsf{d}x^2}\right\| < \infty, \quad \begin{array}{ll} \phi(0) & = e^{i2\pi\alpha}\phi(2\pi) \\ \phi'(0) & = e^{i2\pi\alpha}\phi'(2\pi) \end{array} \right\}$$

- This is a family of self-adjoint operators depending on α
  - Eigenvalues: $(n-\alpha)^2$
  - Eigenfunctions: $\phi_n(x) = e^{i\alpha x}e^{inx}$
- Assuming that the parameter α depends smoothly with time this is unitarily equivalent to:

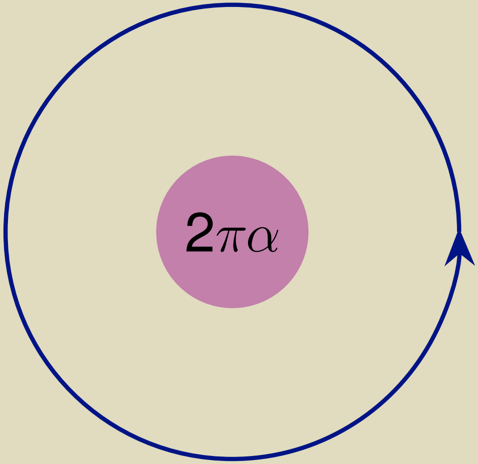
$$H(t) = \left[i\frac{\mathsf{d}}{\mathsf{d}x} - \alpha(t)\right]^2 + \dot{\alpha}(t)x$$

$\mathcal{D}_0 =$ “Periodic Boundary Conditions”

Example: Varying Quasiperiodic Boundary Conditions

$$i\frac{\mathrm{d}}{\mathrm{d}t}\Psi = [i\frac{\mathrm{d}}{\mathrm{d}\theta} - \alpha]^2\Psi + \theta\dot{\alpha}\Psi$$

Quantum Faraday Law



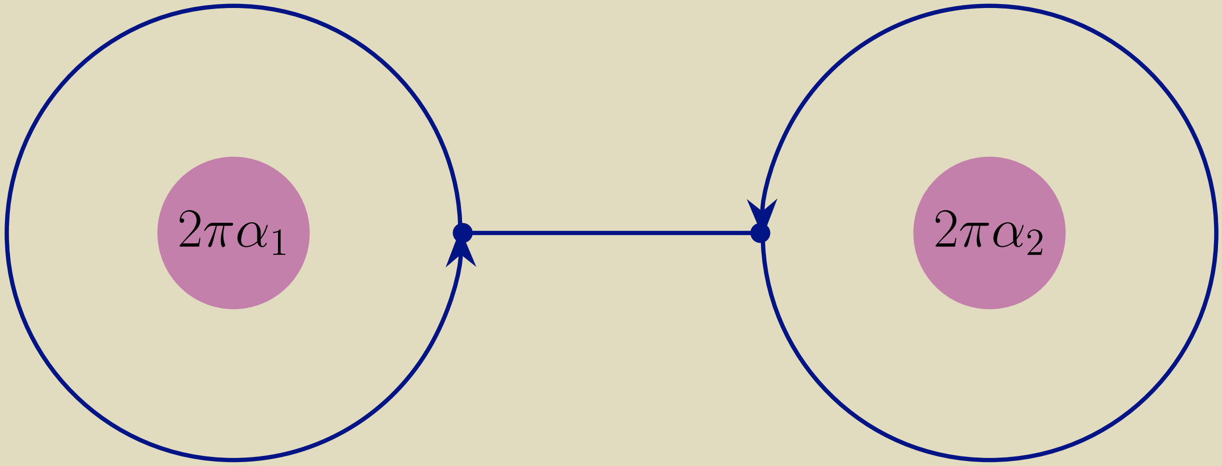
Particle Moving in a circular wire

Magnetic flux of intensity $2\pi\alpha$

Theorem [Balmaseda, Ibort, P.P]:

The one-dimensional Laplacian with quasi-periodic boundary conditions is approximately controllable.

- One can generalise this to more general planar graphs.
- There is a correspondence between families of self-adjoint extensions of the Laplacian and Magnetic Laplacians on graphs.



References

- M. Barbero-Liñán, A. Ibort, JMPP. Boundary dynamics and topology change in quantum mechanics.
  Int. J. Geom. Methods in Modern Phys. **12** 156011, (2015).
- A. Ibort, JMPP. Quantum control and representation theory. J. Phys. A: Math. Theor. **42** 205301, (2009).
- A.P. Balachandran, G. Bimonte, G. Marmo, A. Simoni. Nucl. Phys. B **446** 299-214, (1995).