Solutions of the Laplacian flow and coflow of a Locally Conformal Parallel G_2-structure

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Outline

1. Motivation and G_2 background.
2. Laplacian flow and coflow.
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The holonomy group of a linear connection

Suppose M^m simply connected.

Let ∇ be a linear connection on M, $p \in M$ and $\gamma\colon [0,1] \longrightarrow M$ a curve such that $\gamma(0) = p = \gamma(1)$.

The parallel transport along γ defines an endomorphism: $P_\gamma\colon T_pM \longrightarrow T_pM$.

The holonomy group $Hol_p(\nabla)$ of ∇ based at p: $\{P_\gamma\} \subset GL(m, \mathbb{R})$

For all $p, q \in M$, $Hol_p(\nabla)$ is conjugated to $Hol_q(\nabla)$: the holonomy group $Hol(\nabla)$ of ∇.

In particular, if g is a Riemannian metric and $\nabla = \nabla^g$ is the Levi-Civita connection, then $Hol(\nabla^g) \subset \mathrm{O}(m)$ ($\mathrm{SO}(m)$ if M is orientable).

Riemannian holonomy

[Berger'55]: Some of the possible holonomy groups of a Riemannian, simply connected, irreducible and nonsymmetric (M, g) are:

$Hol(\nabla^g) \subseteq \mathrm{SU}(n)$ in dimension $m = 2n$ (Calabi-Yau);

$Hol(\nabla^g) \subseteq \mathrm{G}_2$ in dimension $m = 7$;

$Hol(\nabla^g) \subseteq \mathrm{Spin}(7)$ in dimension $m = 8$.

In these cases, the metric g is Ricci flat.

Question: "Are there Riemannian metrics g with special $Hol(\nabla^g)$?":
Affirmative answers: Bryant (1987), Bryant-Salamon (1989), Joyce (1996), Hitchin (2001), Kovalev (2003)...

G_2-structures

A 7-dimensional manifold M has a G_2-structure $\iff$ exists a global 3-form (fundamental form) $\sigma \in \Omega^3(M)$ having the local expression

$$\sigma = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}$$

where $\{e^1, \dots, e^7\}$ is a local coframe and e^{ij} stands for $e^i \wedge e^j$.

- The metric induced by σ:

$$g(X,Y)vol = \frac{1}{6}\iota_X\sigma \wedge \iota_Y\sigma \wedge \sigma$$

for any $X, Y \in \mathfrak{X}(M)$.

- Volume form:

$$vol = e^{1234567}$$

- The 4-form $\psi = *\sigma$:

$$\psi = *\sigma = e^{1234} + e^{1256} + e^{3456} + e^{1367} + e^{1457} + e^{2357} - e^{2467}$$

Fernández-Gray classification

[Fernández-Gray'82]

$$\nabla^{LC}\sigma \in \Omega^1 \otimes \Omega^3_7 = X.$$

Under the action of the group G_2, X can be decomposed into 4 irreducible components

$$X = X_1 \oplus X_2 \oplus X_3 \oplus X_4$$

Therefore, there exist 16 different classes of G_2-structures, called the Fernández-Gray classes. Some examples:

Type	Class	Conditions
Parallel	$\mathcal{P}$	$d\sigma = d*\sigma = 0$
Calibrated	X_2	$d\sigma = 0$
Cocalibrated	$X_1 \oplus X_3$	$d(*\sigma) = 0$
Locally Conformal Parallel	X_4	$d\sigma = 3\tau_1 \wedge \sigma$, $\mathrm{d}(*\sigma) = 4\tau_1 \wedge (*\sigma)$

$$\tau_1 \in \Omega^1(M).$$

Laplacian flow of a G_2-structure

[Bryant'06] introduced the Laplacian flow (LF) $\boxed{\frac{d}{dt}\sigma_t = \Delta_t \sigma_t}$

where Δ is the Hodge Laplacian $\Delta = \delta d + d\delta$, with $\delta : \Omega^p \longrightarrow \Omega^{p-1}$ such that $\delta = (-1)^p * d *$.

Properties:

- $\begin{cases} \sigma_t \text{ solution of (LF)} \\ \sigma_0 \text{ calibrated } (d\sigma_0 = 0) \end{cases} \Longrightarrow \sigma_t$ also calibrated.
- Laplacian flow can be considerate as the gradient flow of the Hitchin's volume functional.
- If there exist solution, it converges to a torsion-free (parallel) G_2-structure, i.e. $\mathrm{Hol}(\nabla^{g_\infty}) = \mathrm{G}_2$.

Results:

- [Bryant-Xu'11] Short time existence and uniqueness of solution for compact manifolds (DeTurck's Trick).
- [Lotay-Wei'15] Long time existence of solution starting near a torsion free structure.
- [Fernández-Fino-Manero'16] First examples of long time existence of solution.

Laplacian coflow of a G_2-structure

[Karigiannis-McKay-Tsui'12] introduced the Laplacian coflow (LcF)

$$\boxed{\frac{d}{dt}\psi_t = -\Delta_t \psi_t} \text{ where } \psi = *\sigma.$$

Properties:

- $\begin{cases} \sigma_t \text{ solution of (LcF)} \\ \sigma_0 \text{ cocalibrated } (d * \sigma_0 = 0) \end{cases} \implies \sigma_t$ also cocalibrated.
- If there exist solution, it converges to a torsion-free (parallel) G_2-structure, i.e. $\mathrm{Hol}(\nabla^{g_\infty}) = \mathrm{G}_2$.

Results:

- Short time existence and uniqueness of solution is not known.
- [Grigorian'13] Introduced the modified Laplacian coflow and proved short time existence and uniqueness of solution.
- [Bagaglini-Fernández-Fino'17] First examples of long time existence of solution for coflow and modified coflow.

Laplacian flow and coflow of a Locally Conformal Parallel G_2-structure

In [Manero-Otal-V.'17] we study the (LF) and (LcF) starting from an LCP G_2-structure.

Questions:

- There exists solution for these flows?
- The solutions remain LCP?
- Is there any correspondence between solutions?

We want to solve:

$$\begin{cases} \dfrac{d}{dt}\sigma_t = \Delta_t \sigma_t, \\ \sigma_0 = \sigma, \\ d\sigma_t = 3\,\tau_1(t) \wedge \sigma_t, \\ d *_t \sigma_t = 4\,\tau_1(t) \wedge *_t\sigma_t. \end{cases} \qquad \begin{cases} \dfrac{d}{dt}\psi_t = -\Delta_t \psi_t, \\ \psi_0 = \psi, \\ d\psi_t = 4\,\tau_1(t) \wedge \psi_t, \\ d *_t \psi_t = 3\,\tau_1(t) \wedge *_t\psi_t. \end{cases}$$

Let us study them independently for a particular ansatz.

Our ansatz

Suppose that $\{e^1, \dots, e^7\}$ is an orthonormal local coframe in a G_2-manifold (M^7, σ).

Defomation: Consider a time-dependent coframe $\{x^1(t), \dots, x^7(t)\}$

$$x^k(t) = h_k(t)e^k,$$

with $h_k(t)$ differentiable functions, $h_k(t) \neq 0$ and $h_k(0) = 1$.

Notation: $x^k \equiv x^k(t)$.

We define a one-parameter family of G_2-structures on M as:

$$\sigma_t = x^{127} + x^{347} + x^{567} + x^{135} - x^{146} - x^{236} - x^{245},$$
$$\psi_t = x^{3456} + x^{1256} + x^{1234} - x^{2467} + x^{2357} + x^{1457} + x^{1367}.$$

In terms of the basis $\{e^1, \dots, e^7\}$:

$$\sigma_t = h_{127}e^{127} + h_{347}e^{347} + h_{567}e^{567} + h_{135}e^{135} - h_{146}e^{146} - h_{236}e^{236} - h_{245}e^{245},$$

$$\psi_t = h_{3456}e^{3456} + h_{1256}e^{1256} + h_{1234}e^{1234} - h_{2467}e^{2467} + h_{2357}e^{2357} + h_{1457}e^{1457} + h_{1367}e^{1367},$$

where h_{ijk} stands for the product $h_i(t)h_j(t)h_k(t)$.

LCP flow: Solving $\frac{d}{dt}\sigma_t = \Delta_t \sigma_t$

Our ansatz:

- $x^k = h_k(t) e^k$, ($h_k(t)$ are the unknowns!!)
- $\sigma_t = x^{127} + x^{347} + x^{567} + x^{135} - x^{146} - x^{236} - x^{245}$.
- $\{e^1, \ldots, e^7\}$ is orthonormal.

Direct computations:

$$\frac{d}{dt}\sigma_t = \sum_{(i,j,k)\in\mathcal{I}} \left(\frac{h_i'}{h_i} + \frac{h_j'}{h_j} + \frac{h_k'}{h_k}\right) x^{ijk} - \sum_{(i,j,k)\in\mathcal{J}} \left(\frac{h_i'}{h_i} + \frac{h_j'}{h_j} + \frac{h_k'}{h_k}\right) x^{ijk},$$

where $\mathcal{I} = \{(127), (135), (347), (567)\}$ and $\mathcal{J} = \{(146), (236), (245)\}$.

- Now, σ_t solves the evolution equation for the 3-form, if and only if Δ has the following expression:

$$\Delta_t \sigma_t = \sum_{(i,j,k)\in\mathcal{I}} \Delta_{ijk} x^{ijk} - \sum_{(i,j,k)\in\mathcal{J}} \Delta_{ijk} x^{ijk},$$

  where

$$\Delta_{ijk} = \frac{h_i'}{h_i} + \frac{h_j'}{h_j} + \frac{h_k'}{h_k}, \quad (i,j,k) \in \mathcal{I} \cup \mathcal{J}.$$

- Moreover:

$$\Delta_{abc} = \Delta_{pqr} \Rightarrow h_a h_b h_c = h_p h_q h_r.$$

Example

Consider a solvmanifold (compact quotient of solvable Lie group by lattice of maximal rank, $M = G/\Gamma$) whose Lie algebra is defined by:

- Structure equations in terms of basis $\{e^1, \dots, e^7\}$:

$$\mathfrak{cp}_1 = (-e^{17}, -e^{27}, -e^{37}, -e^{47}, -e^{57}, -e^{67}, 0) \quad (\rightsquigarrow de^1 = -e^1 \wedge e^7)$$

- (invariant) LCP G_2-structure:

$$\sigma_0 = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}.$$

- In terms of basis $x^k = h_k(t)e^k$:

$$\mathfrak{cp}_1 = \left(-\frac{1}{h_7}x^{17}, -\frac{1}{h_7}x^{27}, -\frac{1}{h_7}x^{37}, -\frac{1}{h_7}x^{47}, -\frac{1}{h_7}x^{57}, -\frac{1}{h_7}x^{67}, 0\right).$$

- Family of G_2-structures: (We do not know if they are LCP!!)

$$\sigma_t = x^{127} + x^{347} + x^{567} + x^{135} - x^{146} - x^{236} - x^{245}.$$

- Laplacian:

$$\Delta_t \sigma_t = -\frac{1}{h_7^2}\left[8\left(x^{127} + x^{347} + x^{567}\right) + 9\left(x^{135} - x^{146} - x^{236} - x^{245}\right)\right].$$

Example

Laplacian:

$$\Delta_t \sigma_t = -\frac{1}{h_7^2}\left[8\,\underbrace{(x^{127}+x^{347}+x^{567})}_{\text{Group I}} + 9\,(x^{135}-x^{146}-x^{236}-x^{245})\right].$$

Laplacian:

$$\Delta_t \sigma_t = -\frac{1}{h_7^2}\left[8\,\underbrace{(x^{127}+x^{347}+x^{567})}_{\text{Group I}} + 9\,\underbrace{(x^{135}-x^{146}-x^{236}-x^{245})}_{\text{Group II}}\right].$$

Thus:

- $\Delta_{127} = \Delta_{347} = \Delta_{567} = \dfrac{-8}{h_7^2} \Longrightarrow h_1 h_2 h_7 = h_3 h_4 h_7 = h_5 h_6 h_7.$
- $\Delta_{135} = \Delta_{146} = \Delta_{236} = \Delta_{245} = \dfrac{-9}{h_7^2} \Longrightarrow$
  $h_1 h_3 h_5 = h_1 h_4 h_6 = h_2 h_3 h_6 = h_2 h_4 h_5.$

Solving the blue system: $h_1 = h_2 = h_3 = h_4 = h_5 = h_6 = h(t)$.

The evolution equation is equivalent to the system

Example

The evolution equation is equivalent to the system

$$\begin{cases} \frac{-2}{h_7^2} = \frac{h_7'}{h_7} \\ \frac{-3}{h_7^2} = \frac{h'}{h} \end{cases} \xRightarrow{\text{Solution:}} h(t) = (1-4\,t)^{3/4} \text{ and } h_7(t) = (1-4\,t)^{1/2}.$$

Conclusion:

$$\sigma_t = (1-4t)^2(e^{127}+e^{347}+e^{567}) + (1-4t)^{\frac{9}{4}}(e^{135}-e^{146}-e^{236}-e^{245})$$

for $t \in (-\infty, \frac{1}{4})$ solves the evolution equation $\frac{d}{dt}\sigma_t = \Delta_t \sigma_t$.

Moreover, can be checked that it remains LCP for any t:

$$\begin{cases} d\sigma_t = 3\tau_1(t)\wedge\sigma_t, \\ d*\sigma_t = 4\tau_1(t)\wedge(*\sigma_t), \end{cases} \qquad \text{with } \tau_1(t) = e^7.$$

Therefore, it is a solution for the LCP Laplacian flow

Finally, observe that the metric g_t remains Einstein for all $t \in (-\infty, \frac{1}{4})$ since

$$Ric(g_t) = -\frac{6}{1-4t} g_t.$$

LCP coflow: Solving $\frac{d}{dt}\psi_t = -\Delta_t\psi_t$

Similarly as the LCP flow, using $x^k = h_k(t)e^k$, and taking into account that

$$\psi_t = x^{3456} + x^{1256} + x^{1234} - x^{2467} + x^{2357} + x^{1457} + x^{1367} :$$

$$\frac{d}{dt}\psi_t = \sum_{(l,m,n,o)\in\mathcal{K}} \left(\frac{h_l'}{h_l} + \frac{h_m'}{h_m} + \frac{h_n'}{h_n} + \frac{h_o'}{h_o}\right) x^{lmno} - \left(\frac{h_2'}{h_2} + \frac{h_4'}{h_4} + \frac{h_6'}{h_6} + \frac{h_7'}{h_7}\right) x^{2467},$$

where $\mathcal{K} = \{(1234), (1256), (1367), (1457), (2357), (3456)\}$.

- Now, σ_t solves the evolution equation for the 4-form, if and only if Δ has the following expression:

$$\Delta_t\psi_t = \sum_{(l,m,n,o)\in\mathcal{K}} \Delta_{lmno}x^{lmno} - \Delta_{2467}x^{2467},$$

  where

$$\Delta_{lmno} = \frac{h_l'}{h_l} + \frac{h_m'}{h_m} + \frac{h_n'}{h_n} + \frac{h_o'}{h_o}, \quad (l,m,n,o) \in \mathcal{K} \cup (2467).$$

- Moreover:

$$\Delta_{lmno} = \Delta_{pqrs} \Rightarrow h_l h_m h_n h_o = h_p h_q h_r h_s.$$

Solvmanifolds with an LCP G_2-structure

[Chiossi-Fino'06] Obtained a family of solvmanifolds endowed with an LCP G_2-structure as a rank one solvable extension of 6-dim nilpotent Lie groups endowed with SU(3)-structure.

$$\mathfrak{cp}_1 = (-e^{17}, -e^{27}, -e^{37}, -e^{47}, -e^{57}, -e^{67}, 0);$$

$$\mathfrak{cp}_2 = \left(-\frac{4}{3}e^{17} + \frac{2}{3}e^{36}, -e^{27}, -\frac{2}{3}e^{37}, -e^{47}, -e^{57}, -\frac{2}{3}e^{67}, 0\right);$$

$$\mathfrak{cp}_3 = \left(-\frac{3}{2}e^{17} + \frac{1}{2}e^{36} + \frac{1}{2}e^{45}, -e^{27}, -\frac{3}{4}e^{37}, -\frac{3}{4}e^{47}, -\frac{3}{4}e^{57}, -\frac{3}{4}e^{67}, 0\right);$$

$$\mathfrak{cp}_4 = \left(-\frac{7}{5}e^{17} + \frac{2}{5}e^{36} + \frac{2}{5}e^{45}, -\frac{6}{5}e^{27} - \frac{2}{5}e^{46}, -\frac{4}{5}e^{37}, -\frac{3}{5}e^{47}, -\frac{4}{5}e^{57}, -\frac{3}{5}e^{67}, 0\right);$$

$$\mathfrak{cp}_5 = \left(-\frac{5}{4}e^{17} + \frac{1}{2}e^{45}, -\frac{5}{4}e^{27} - \frac{1}{2}e^{45}, -e^{37}, -\frac{1}{2}e^{47}, -\frac{3}{4}e^{57}, -\frac{3}{4}e^{67}, 0\right);$$

$$\mathfrak{cp}_6 = \left(-\frac{4}{3}e^{17} + \frac{1}{3}e^{36} + \frac{1}{3}e^{45}, -\frac{4}{3}e^{27} + \frac{1}{3}e^{35} - \frac{1}{3}e^{46}, -\frac{2}{3}e^{37}, -\frac{2}{3}e^{47}, -\frac{2}{3}e^{57}, -\frac{2}{3}e^{67}, 0\right);$$

$$\mathfrak{cp}_7 = \left(-\frac{6}{5}e^{17} + \frac{2}{5}e^{36}, -\frac{3}{5}e^{27}, -\frac{3}{5}e^{37}, \frac{2}{5}e^{26} - \frac{6}{5}e^{47}, \frac{2}{5}e^{23} - \frac{6}{5}e^{57}, -\frac{3}{5}e^{67}, 0\right);$$

Main result: Every 7-dimensional rank-one solvable extension of a nilpotent Lie group with a Locally Conformal Parallel G_2-structure admits

- a long time LCP solution to the Laplacian flow.
- a long time LCP solution to the Laplacian coflow.

More solutions to the LCP Laplacian flow & coflow

For the rest of the cases $\mathfrak{cp}_i$ with $i = \{2, \ldots, 7\}$, we consider a basis $\{x^1, \ldots, x^7\}$ of 1-forms given by $x^k = h_k(t)e^k$, and a particular type of functions $h_k(t)$, inspired by the previous example:

$$\text{flow} \Rightarrow h_k(t) = (1 - \alpha t)^{\beta_k}, \quad \text{coflow} \Rightarrow h_k(t) = (1 - \gamma t)^{\delta_k}.$$

$\mathfrak{cp}_i$	α	$(\beta_1, \ldots, \beta_7)$	γ	$(\delta_1, \ldots, \delta_7)$
$\mathfrak{cp}_1$	4	$\left(\frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{1}{2}\right)$	-6	$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{2}\right)$
$\mathfrak{cp}_2$	$\frac{10}{3}$	$\left(\frac{9}{10}, \frac{4}{5}, \frac{7}{10}, \frac{4}{5}, \frac{4}{5}, \frac{7}{10}, \frac{1}{2}\right)$	$\frac{-16}{3}$	$\left(\frac{1}{4}, \frac{5}{16}, \frac{3}{8}, \frac{5}{16}, \frac{5}{16}, \frac{3}{8}, \frac{1}{2}\right)$
$\mathfrak{cp}_3$	3	$\left(1, \frac{5}{6}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{1}{2}\right)$	-5	$\left(\frac{1}{5}, \frac{3}{10}, \frac{7}{20}, \frac{7}{20}, \frac{7}{20}, \frac{7}{20}, \frac{1}{2}\right)$
$\mathfrak{cp}_4$	$\frac{14}{5}$	$\left(1, \frac{13}{14}, \frac{11}{14}, \frac{5}{7}, \frac{11}{14}, \frac{5}{7}, \frac{1}{2}\right)$	$-\frac{24}{5}$	$\left(\frac{5}{24}, \frac{1}{4}, \frac{1}{3}, \frac{3}{8}, \frac{1}{3}, \frac{3}{8}, \frac{1}{2}\right)$
$\mathfrak{cp}_5$	3	$\left(\frac{11}{12}, \frac{11}{12}, \frac{5}{6}, \frac{2}{3}, \frac{3}{4}, \frac{3}{4}, \frac{1}{2}\right)$	-5	$\left(\frac{1}{4}, \frac{1}{4}, \frac{3}{10}, \frac{2}{5}, \frac{7}{20}, \frac{7}{20}, \frac{1}{2}\right)$
$\mathfrak{cp}_6$	$\frac{8}{3}$	$\left(1, 1, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{1}{2}\right)$	$-\frac{14}{3}$	$\left(\frac{3}{14}, \frac{3}{14}, \frac{5}{14}, \frac{5}{14}, \frac{5}{14}, \frac{5}{14}, \frac{1}{2}\right)$
$\mathfrak{cp}_7$	$\frac{14}{5}$	$\left(\frac{13}{14}, \frac{10}{14}, \frac{10}{14}, \frac{13}{14}, \frac{13}{14}, \frac{10}{14}, \frac{1}{2}\right)$	$-\frac{24}{5}$	$\left(\frac{1}{4}, \frac{3}{8}, \frac{3}{8}, \frac{1}{4}, \frac{1}{4}, \frac{3}{8}, \frac{1}{2}\right)$

Relation Theorem

The founded solutions for the LCP flow and the LCP coflow are related, as the following Theorem shows:

Theorem

Let σ_t and $\widetilde{\sigma}_t$ be two different families of G_2 structures on $\mathfrak{cp}_i$ with $i = 1, \ldots, 7$, where

$$h_k(t) = (1-\alpha\, t)^{\beta_k}, \quad \beta_7 = \frac{1}{2}, \quad \text{and} \quad \widetilde{h}_k(t) = (1-\gamma\, t)^{\delta_k}, \quad \delta_7 = \frac{1}{2}.$$

If the defining parameters of the functions $h_i(t)$ and $\widetilde{h}_i(t)$ are related by:

$$\boxed{\gamma = \alpha\left(\frac{2-\sum_{j=1}^{7}\beta_j}{2}\right), \delta_k = \frac{1}{2} + \frac{1-2\beta_k}{-2+\sum_{j=1}^{7}\beta_j} \quad \text{for } k \in \{1,\ldots,7\}.}$$

Then:

(i) *σ_t is LCP if and only if $\widetilde{\sigma}_t$ is LCP.*

(ii) *σ_t solves the Laplacian flow if and only if $\widetilde{\psi}_t = *_{\widetilde{\sigma}}\widetilde{\sigma}_t$ solves the Laplacian coflow.*

Open problems

We have obtained some long-time solutions to the LCP flow and coflow.

Open problems:

- Study short time-existence and uniqueness of solution.
- Study the behavior of limit of solutions. Are they parallel structures?

Thank you!!