# Genus Integration, Abelianization and Extended Monodromy

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XXVII International Fall Workshop on Geometry and Physics
Sevilla, September 2018

This talk is based on:

- Ivan Contreras & RLF, "Genus Integration, Abelianization and Extended Monodromy", arXiv:1805.12043.

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**Plan:**

1. Genus integration
2. Abelianization of Lie algebroids and Lie groupoids
3. Extended monodromy

# The canonical integration

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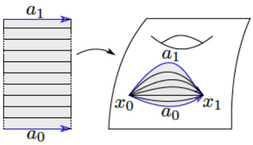

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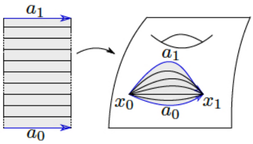


Topological groupoid with structure maps:

- source: $\mathbf{s}([a]) = p(a(0))$;
- target: $\mathbf{t}([a]) = p(a(1))$;
- product: $[a] \cdot [b] = [a \circ b]$;

# Monodromy

For each $x \in M$:

- isotropy Lie algebra: $\mathfrak{g}_x = \ker \rho_x$;
- orbit: $\mathcal{O}_x \subset M$ such that $T_y\mathcal{O} = \mathrm{Im}\,\rho_y$.

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## Theorem (Crainic & RLF, 2003)

*The following statements are equivalent:*

(i) *$A$ integrates to some Lie groupoid;*

(ii) *$\mathit{\Pi}_1(A)$ is a Lie groupoid;*

(iii) *The monodromy groups $\mathcal{N}_x = \mathrm{Im}\,\partial_x$ are uniformly discrete.*

# Genus integration

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## Definition

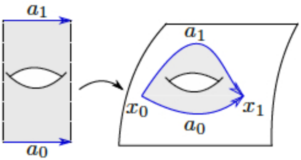
An $A$-**homology** between $A$-paths $a_0$ and $a_1$ is a Lie algebroid map

$$h : T\Sigma \to A,$$

where $\Sigma$ is a compact surface with connected boundary $\partial\Sigma$ such that

$$h|_{T(\partial\Sigma)} = a_0 \circ a_1^{-1}.$$

We denote by $[[a]]$ the $A$-**homology class** of the $A$-path $a$.



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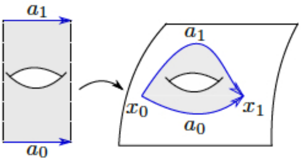


**Rmk.** The genus of $\Sigma$ is not fixed.

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**Basic questions:**

- What is the meaning of this genus integration?
- When is $\mathcal{H}_1(A)$ smooth?
- If $\mathcal{H}_1(A)$ is smooth, what is its Lie algebroid?

# Abelianization of Lie algebroids

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**Examples**

1) Abelian Lie algebras $A = \mathfrak{g}$;
2) Tangent bundle $A = TM$ (more generally, foliations $A = T\mathcal{F}$);
3) Regular Poisson structures $A = T^*M$;
4) Infinitesimal actions with abelian isotropy $A = \mathfrak{g} \ltimes M$.

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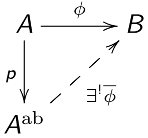
## Definition

The **abelianization** of $A \to M$ is an abelian Lie algebroid $A^{\mathrm{ab}} \to M$ together with a surjective morphism $p : A \to A^{\mathrm{ab}}$ such that: for any *abelian* Lie algebroid $B \to N$ and any morphism $\phi : A \to B$:



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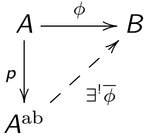


**Remarks.**

- $A$ and $A^{\mathrm{ab}}$ have the same orbit foliation;
- For any $x \in M$, $[\mathfrak{g}_x, \mathfrak{g}_x] \subset \ker p_x$.

# Abelianization of Lie algebroids

**Examples**

1) The abelianization of a transitive Lie algebroid $A \to M$ is:

$$A^{\mathrm{ab}} = A/[\mathfrak{g}, \mathfrak{g}], \quad \text{where} \quad [\mathfrak{g}, \mathfrak{g}]_x := [\mathfrak{g}_x, \mathfrak{g}_x].$$

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2) The bundle of Lie algebras $A = \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ with Lie bracket $[e_1, e_2] = xe_2$ has abelianization the trivial line bundle:

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**Open problem:** Characterize the Lie algebroids which have an abelianization.

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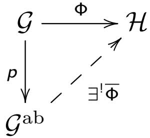


**Rmk.** The terms "groupoid" and "groupoid morphism" should be interpreted in one of the categories **Sets**, **Top** or **Man**.

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However, when $G$ is 1-connected, we have $\overline{(G,G)} = (G,G)$ and $G^{\mathrm{ab}}$ is the 1-connected Lie group integrating $\mathfrak{g}^{\mathrm{ab}}$.

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## Theorem

*For any Lie algebroid $A \to M$:*

$$\mathcal{H}_1(A) = \Pi_1(A)/(\Pi_1(A), \Pi_1(A)),$$

*where $(\Pi_1(A), \Pi_1(A)) = \bigcup_{x \in M} (\Pi_1(A)_x, \Pi_1(A)_x)$ is the group bundle formed by the isotropies of $\Pi_1(A)$.*

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**Remark.** $\mathcal{H}_1(A)$ need to be source 1-connected!

# Examples

1) For a Lie algebra $A = \mathfrak{g}$:
   - $\Pi_1(\mathfrak{g}) = \tilde{G}$ is smooth);
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5) For the bundle of Lie algebras $A = \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$, w/ $[e_1, e_2] = xe_2$:
   - $\Pi_1(A) = \mathbb{R} \ltimes \mathbb{R}^2 \to R$ with product $(x, a_1, b_1) \cdot (x, a_2, b_2) = (x, a_1 a_2, a_1^x b_2 + b_1)$;
   - $\Pi_1(A)^{\mathrm{ab}} = \mathbb{R} \ltimes \mathbb{R} \to R$;
   - $\mathcal{H}_1(A) = \Pi_1(A)^{\mathrm{ab}} \cup \{0\} \times \mathbb{R}^2$ is not smooth;

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Consider the restriction $A_{\mathcal{O}}$ to an orbit $\mathcal{O}$, and choose a splitting $\sigma : T\mathcal{O} \to A^{\mathrm{ab}}_{\mathcal{O}}$ for the short exact sequence of the abelianization:

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Then:

- curvature 2-form $\Omega \in \Omega^2(\mathcal{O}, \mathfrak{g}_{\mathcal{O}}^{\mathrm{ab}})$:

$$\Omega(X, Y) := [\sigma(X), \sigma(Y)] - \sigma([X, Y]).$$

- flat connection $\nabla$ on the bundle $\mathfrak{g}_{\mathcal{O}}^{\mathrm{ab}} \to \mathcal{O}$:

$$\nabla_X \alpha := [\sigma(X), \alpha].$$

**Remark.** Two different splittings induce the same connection and the same curvature 2-form.

# Extended Monodromy

Let $q : \tilde{\mathcal{O}}^h \to \mathcal{O}$ be the holonomy cover of $\mathcal{O}$ relative to $\nabla$, so $q^*\mathfrak{g}_{\mathcal{O}}^{\mathrm{ab}} \to \tilde{\mathcal{O}}^h$ is trivial with a canonical trivialization.

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## Definition

The **extended monodromy homomorphism** at $x \in M$ is the homomorphism of abelian groups:

$$\partial_x^{\mathrm{ext}} : H_2(\tilde{\mathcal{O}}^h) \to \Pi_1(\mathfrak{g}_x^{\mathrm{ab}}), \quad [\gamma] \mapsto \exp\left(\int_\gamma q^*\Omega\right).$$

Its image $\mathcal{N}_x^{\mathrm{ext}}(A) \subset \Pi_1(\mathfrak{g}_x^{\mathrm{ab}})$ is the **extended monodromy group** at $x$.

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## Proposition

*There is a commutative diagram:*

$$\begin{array}{ccc} \pi_2(\mathcal{O}, x) & \xrightarrow{\partial_x} & \mathcal{G}(\mathfrak{g}_x) \\ \downarrow & & \downarrow \\ H_2(\tilde{\mathcal{O}}^h) & \xrightarrow[\partial_x^{\mathrm{ext}}]{} & \mathcal{G}(\mathfrak{g}_x^{\mathrm{ab}}) \end{array}$$

# Extended Monodromy

### Theorem

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(b) *the extended monodromy groups are discrete;*

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### Corollary

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### Corollary

*If a Lie algebroid $A \to M$ admits a proper integration $\mathcal{G} \rightrightarrows M$ then its extended monodromy groups are all discrete.*

# Closing Remarks/Open Problems

- Determine the obstructions to the existence of the abelianization $A^{\mathrm{ab}}$;
- Determine the obstructions to smoothness of $\mathcal{H}_1(A)$ in the non-transitive case;
- Poisson geometry: does $\mathcal{H}_1(T^*M)$ have extra geometric structure?
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THANK YOU!