Contact Dual Pairs

Alfonso Giuseppe Tortorella

KU LEUVEN

(joint work with: A. Blaga, M. A. Salazar & C. Vizman)

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Introduction

Symplectic dual pairs, tracing back to [LIE, 1890], have their modern origin in

- group representations arising from quantum mechanics [HOWE],
- the local structure of Poisson manifolds [WEINSTEIN].

They play a relevant role in Poisson geometry and Hamiltonian dynamics:

- Morita equivalence of Poisson manifolds,
- superintegrable Hamiltonian systems,
- moment maps and reduction theory.

The close analogy existing between symplectic/Poisson and contact/Jacobi geometry makes it pretty natural to wonder:

what, if any, is the contact analogue of symplectic dual pairs?

Outline

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Contact Manifolds

A *pre-contact structure* on a manifold M is equivalently given by:

- a *pre-contact distribution*, i.e. an hyperplane distribution $\mathcal{H} \subset TM$, or
- a *pre-contact form*, i.e. a 1-form ϑ on M with values in a line bundle L,

with $L = TM/\mathcal{H}$ and $\vartheta(X) = X \bmod \mathcal{H}$, or in the opposite direction $\mathcal{H} = \ker \vartheta$.

If $\mathcal{H}$ is *maximally non-integrable*, i.e. the *curvature* 2*-form* $\omega_{\mathcal{H}} \in \Gamma(\wedge^2 \mathcal{H}^* \otimes L)$, with $\omega_{\mathcal{H}}(X, Y) := \vartheta[X, Y]$, is non-degenerate, then $\mathcal{H}$ and ϑ are said to be *contact*.

A *contact structure* is equivalently given by a contact distribution $\mathcal{H}$ or a contact form ϑ. A *contact manifold* is a manifold equipped with a contact structure.

In the *coorientable case*, i.e. when $L \longrightarrow M$ admits a global frame, one gets that $\vartheta \in \Omega^1(M)$, $\omega_{\mathcal{H}} = (\mathrm{d}\vartheta)|_{\mathcal{H}}$, and the maximal non-integrability of $\mathcal{H}$ reduces to:

$$(\mathrm{d}\vartheta)^n \wedge \vartheta \ \text{ is a volume form on } M^{2n+1}.$$

A *contactomorphism* $(M, \mathcal{H}) \to (M', \mathcal{H}')$ is a diffeo $\underline{\varphi} : M \to M'$ s.t. $\underline{\varphi}_* \mathcal{H} = \mathcal{H}'$, or equivalently a line bundle isomorphism $\widehat{\varphi} : L \to L'$ s.t. $\widehat{\varphi}^* \vartheta' = \vartheta$.

By symplectization (and projectivization) the contact category is equivalent to the category of equivariant symplectomorphisms between $\mathbb{R}^\times$-principal bundles equipped with (degree 1) homogeneous symplectic structures.

The Lie algebra of *infinitesimal contactomorphisms* (or *contact vector fields*) is

$$\mathfrak{cont}(M, \mathcal{H}) = \{X \in \mathfrak{X}(M) : [X, \Gamma(\mathcal{H})] \subset \Gamma(\mathcal{H})\}.$$

The $\mathbb{R}$-linear isomorphism $\Gamma(L) \to \mathfrak{cont}(M, \mathcal{H})$, $\lambda \mapsto \mathcal{X}_\lambda$, uniquely determined by $\vartheta(\mathcal{X}_\lambda) = \lambda$, is additionally a 1st order differential operator from L to TM.

The skew-symmetric 1st order bi-differential operator satisfying Jacobi identity

$$\{-, -\}_\mathcal{H} : \Gamma(L) \times \Gamma(L) \to \Gamma(L), \ (\lambda, \mu) \mapsto \{\lambda, \mu\}_\mathcal{H} := \vartheta[\mathcal{X}_\lambda, \mathcal{X}_\mu],$$

is the associated *Jacobi structure* and fully encodes the contact structure.

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Jacobi manifolds

A *Jacobi bundle* [MARLE] is a line bundle $L \to M$ with a *Jacobi structure*, i.e. a Lie bracket $\mathrm{J} = \{-,-\} : \Gamma(L) \times \Gamma(L) \to \Gamma(L)$ which is a 1st order bi-differential operator. A *Jacobi manifold* is a manifold with a Jacobi bundle over it.

- In particular, the Poisson structures on a manifold M coincide exactly with those Jacobi structures $\{-,-\}$ on $\mathbb{R}_M := M \times \mathbb{R} \to M$ such that $\{1,-\} = 0$,
- the contact structures coincide with the *non-degenerate* Jacobi structures.

A *Jacobi morphism* $(M_1, L_1, \{-,-\}_1) \to (M_2, L_2, \{-,-\}_2)$ is a *regular* line bundle morphism $\varphi : L_1 \to L_2$ s.t. the pull-back of sections $\varphi^* : \Gamma(L_2) \to \Gamma(L_1)$ satisfies

$$\varphi^*\{\lambda, \mu\}_2 = \{\varphi^*\lambda, \varphi^*\mu\}_1, \quad \forall \lambda, \mu \in \Gamma(L_2).$$

By Poissonization (and projectivization) the Jacobi category is equivalent to the category of equivariant Poisson maps between $\mathbb{R}^\times$-principal bundles equipped with (degree -1) homogeneous Poisson structures.

Locally conformal symplectic manifolds

A *locally conformal symplectic (lcs) structure* on a manifold M consists of a representation ∇ of TM on line bundle $L \to M$ and non-degenerate d_∇-closed $\omega \in \Omega^2(M; L)$. An *lcs manifold* is a manifold equipped with a lcs structure.

The $\mathbb{R}$-linear map $\Gamma(L) \to \mathfrak{X}(M)$, $\lambda \mapsto \mathcal{X}_\lambda$, uniquely determined by $\mathcal{X}_\lambda := \omega^\sharp(d_\nabla \lambda)$, is additionally a 1st order differential operator from L to TM.

The skew-symmetric 1st order bi-differential operator satisfying Jacobi identity

$$\{-,-\}_M : \Gamma(L) \times \Gamma(L) \to \Gamma(L),\ (\lambda, \mu) \mapsto \{\lambda, \mu\}_M := \omega(\mathcal{X}_\lambda, \mathcal{X}_\mu),$$

is the associated *Jacobi structure* and fully encodes the lcs structure.

In the *coorientable case*, an lcs structure reduces to a pair (η, ω), formed by a closed $\eta \in \Omega^1(M)$ and a non-degenerate $\omega \in \Omega^2(M)$, such that

$$d\omega + \omega \wedge \eta = 0. \qquad \text{[VAISMAN]}$$

The Characteristic Distribution of a Jacobi Manifold

Let $(M, L, \{-,-\})$ be a Jacobi manifold. For any $\lambda \in \Gamma(L)$, there is an associated *Hamiltonian vector field* $\mathcal{X}_\lambda \in \mathfrak{X}(M)$ uniquely determined by

$$\{\lambda, f\mu\} = \mathcal{X}_\lambda(f)\mu + f\{\lambda, \mu\}, \qquad \forall \mu \in \Gamma(L),\ f \in C^\infty(M).$$

The *characteristic distribution* of $(M, L, \{-,-\})$ is the singular distribution

$$\mathcal{C} = \mathrm{span}\{\mathcal{X}_\lambda : \lambda \in \Gamma(L)\} \subset TM.$$

If $\mathcal{C} = TM$, then the Jacobi structure is said to be *transitive*.

Characteristic Foliation Theorem [Kirillov '76]

1. A transitive Jacobi manifold is completely characterized as follows:
   - odd-dimensional transitive Jacobi manifolds $\xleftrightarrow{\text{1-1}}$ contact manifolds,
   - even-dimensional transitive Jacobi manifolds $\xleftrightarrow{\text{1-1}}$ lcs manifolds.
2. The characteristic distribution $\mathcal{C}$ is integrable à la Stefan–Sussmann, and each characteristic integral leaf inherits a transitive Jacobi structure.

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Contact Groupoids

Let $\mathcal{G} \rightrightarrows \mathcal{G}_0$ be a Lie groupoid, with structure maps s, t, m, u, i. The duality contact distributions and forms induces a correspondence between:

- contact distributions $\mathcal{H}$ on $\mathcal{G}$ which are *multiplicative*, i.e. $\mathcal{H}$ is a *wide* subgroupoid of the tangent groupoid $T\mathcal{G} \rightrightarrows T\mathcal{G}_0$,
- contact forms $\vartheta \in \Omega^1(\mathcal{G}; t^*L_0)$, with values in a representation $L_0 \to \mathcal{G}_0$ of $\mathcal{G}$, which are *multiplicative*, i.e. for all $(g,h) \in \mathcal{G}^{(2)}$

$$(m^*\vartheta)_{(g,h)} = (\mathrm{pr}_1^*\vartheta)_{(g,h)} + g \cdot (\mathrm{pr}_2^*\vartheta)_{(g,h)}.$$

A *multiplicative contact structure* on a Lie groupoid is equivalently given by a multiplicative contact distribution $\mathcal{H}$ or a multiplicative contact form ϑ. A *contact groupoid* is a Lie groupoid equipped with a multiplicative contact structure.

By symplectization (and projectivization) the category of contact groupoids is equivalent to the category of (degree 1) homogeneous symplectic groupoids.

From Contact Groupoids to Contact Dual Pairs

Let $\mathcal{G} \rightrightarrows \mathcal{G}_0$ be a contact groupoid, $\vartheta \in \Omega^1(\mathcal{G}; t^* L_0)$, $\mathcal{H} = \ker \vartheta$ and $L = T\mathcal{G}/\mathcal{H}$.

- The right and left multiplications induce regular line bundle morphisms $\widehat{s} : L \to L_0$ and $\widehat{t} : L \to L_0$ covering respectively $s : \mathcal{G} \to \mathcal{G}_0$ and $t : \mathcal{G} \to \mathcal{G}_0$.
- There is a unique Jacobi structure $J_0 = \{-,-\}_0$ on $L_0 \to \mathcal{G}_0$ such that the maps in the following diagram are Jacobi morphisms

$$(\mathcal{G}_0, L_0, J_0) \xleftarrow{\ \widehat{s}\ } (\mathcal{G}, \mathcal{H}) \xrightarrow{\ \widehat{t}\ } (\mathcal{G}_0, L_0, -J_0).$$

The following additional properties are satisfied by any contact groupoid:

1. $\mathcal{H}$ is transverse to both $T^s\mathcal{G}$ and $T^t\mathcal{G}$,
2. $\widehat{s}^*\lambda$ and $\widehat{t}^*\mu$ commute wrt $\{-,-\}_{\mathcal{H}}$, for all $\lambda, \mu \in \Gamma(L)$,
3. $\mathcal{H}^s := \mathcal{H} \cap T^s\mathcal{G}$ is the orthogonal complement of $\mathcal{H}^t := \mathcal{H} \cap T^t\mathcal{G}$ wrt $\omega_{\mathcal{H}}$.

so contact groupoids lead us to single out the notion of contact dual pair.

Contact Dual Pairs

Consider a contact manifold $(M, \mathcal{H})$ and a pair of Jacobi morphisms

$$(M_1, L_1, J_1 = \{-,-\}_1) \xleftarrow{\widehat{\varphi}_1} (M, \mathcal{H}) \xrightarrow{\widehat{\varphi}_2} (M_2, L_2, J_2 = \{-,-\}_2). \qquad (\star)$$

DEFINITION The above diagram form a *(Lie–Weinstein) contact dual pair* if:

1. $\mathcal{H}$ is transverse to the fibers of both $\underline{\varphi}_1 : M \to M_1$ and $\underline{\varphi}_2 : M \to M_2$,
2. $\widehat{\varphi}_1^* \lambda_1$ and $\widehat{\varphi}_2^* \lambda_2$ commute wrt $\{-,-\}_{\mathcal{H}}$, for all $\lambda_1 \in \Gamma(L_1)$ and $\lambda_2 \in \Gamma(L_2)$,
3. $\mathcal{H} \cap \ker T\underline{\varphi}_1$ is the orthogonal complement of $\mathcal{H} \cap \ker T\underline{\varphi}_2$ wrt $\omega_{\mathcal{H}}$.

Further $(\star)$ is *full* if $\underline{\varphi}_1 : M \to M_1$ and $\underline{\varphi}_2 : M \to M_2$ are surjective submersions.

- A contact groupoid gives canonically rise to a full contact dual pair.

PROPOSITION The symplectization/Poissonization functor identifies the contact dual pairs with the homogeneous symplectic dual pairs.

Consider a full contact dual pair $(M_1, L_1, J_1) \xleftarrow{\widehat{\varphi}_1} (M, \mathcal{H}) \xrightarrow{\widehat{\varphi}_2} (M_2, L_2, J_2)$, with connected fibers of the underlying maps $\underline{\varphi}_1 : M \to M_1$ and $\underline{\varphi}_2 : M \to M_2$.

The Characteristic Leaf Correspondence Theorem

1. The relation $\underline{\varphi}_1^{-1}(\mathcal{S}_1) = \underline{\varphi}_2^{-1}(\mathcal{S}_2)$ establishes a 1-1 correspondence between the characteristic leaves $\mathcal{S}_1$ of M_1 and the characteristic leaves $\mathcal{S}_2$ of M_2.
2. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be in correspondence. Then $\dim \mathcal{S}_1 \equiv \dim \mathcal{S}_2 \bmod 2$, and
   - if $\mathcal{S}_i$ is odd-dim and $\vartheta_{\mathcal{S}_i}$ is its inherited contact 1-form, then:

     $$\iota_{\mathcal{S}}^* \vartheta = \varphi_1^* \vartheta_{\mathcal{S}_1} + \varphi_2^* \vartheta_{\mathcal{S}_2} \in \Omega^1(\mathcal{S}, L|_{\mathcal{S}}),$$

     where $\underline{\varphi}_1^{-1}(\mathcal{S}_1) = \mathcal{S} = \underline{\varphi}_2^{-1}(\mathcal{S}_2)$ and $\iota_S : L|_{\mathcal{S}} \to L$ is the inclusion map,
   - if $\mathcal{S}_i$ is even-dim and $(L_i|_{\mathcal{S}_i}, \nabla^{\mathcal{S}_i}, \omega_{\mathcal{S}_i})$ is its inherited lcs structure, then

     $$d_\nabla(\iota_{\mathcal{S}}^* \vartheta) = \varphi_1^* \omega_{\mathcal{S}_1} + \varphi_2^* \omega_{\mathcal{S}_2} \in \Omega^2(\mathcal{S}, L|_{\mathcal{S}}),$$

     where ∇ is the representation of $T\mathcal{S}$ on $L|_{\mathcal{S}}$ s.t. $\varphi_1^* \nabla^{\mathcal{S}_1} = \nabla = \varphi_2^* \nabla^{\mathcal{S}_2}$.

Outline

Contact Groupoid Actions

Let M be a contact manifold and $\mathcal{G} \rightrightarrows \mathcal{G}_0$ a contact groupoid. A *(left) action* of $\mathcal{G}$ on M, with *moment* $\mu : M \to \mathcal{G}_0$, is a map $\Phi : \mathcal{G}_s\times_\mu M \to M$, $(g, x) \mapsto g \cdot x$, s.t.

$$\mu(g \cdot x) = t(g), \quad \mu(x) \cdot x = x, \quad (gh) \cdot x = g \cdot (hx).$$

The action is *contact* [ZAMBON, ZHU] is these two equivalent properties hold:

- $(T\Phi)(u, v) \in \mathcal{H}_M \Longleftrightarrow u \in \mathcal{H}_\mathcal{G}$; for all $(u, v) \in T(\mathcal{G}_s\times_\mu M)$, s.t. $v \in \mathcal{H}_M$,
- there is a regular line bundle morphism $\widehat{\mu} : L_M \to L_0$ covering μ (so that $L_M \simeq \mu^* L_0$), and for all $(g, x) \in \mathcal{G}_s\times_\mu M$ the following holds:

$$(\Phi^*\vartheta_M)_{(g,x)} = (\mathrm{pr}_1^*\vartheta_\mathcal{G})_{(g,x)} + g \cdot (\mathrm{pr}_2^*\vartheta_M)_{(g,x)}.$$

- $\mathcal{G}^{(2)} \xrightarrow{m} \mathcal{G}$ is a contact groupoid action of $\mathcal{G}$ on $\mathcal{G}$ with moment $\mathcal{G} \xrightarrow{t} \mathcal{G}_0$.
- a Lie group G acting on M by contactomorphisms gives rise to a moment $\mu : M \to \mathbb{P}(\mathfrak{g}^*)$ and a contact groupoid action of $\mathbb{P}(T^*G) \rightrightarrows \mathbb{P}(\mathfrak{g}^*)$ on M.

Contact Dual Pairs from Contact Reduction

Fix a *free and proper* contact action of a *s-connected* contact groupoid $\mathcal{G} \rightrightarrows \mathcal{G}_0$ on a contact manifold M with moment $\mu : M \to \mathcal{G}_0$. Then one gets that:

- $\mu : M \to \mathcal{G}_0$ is a submersion, with image denoted $\mathcal{G}_0^\mu$, and it lifts to a $\mathcal{G}$-equivariant Jacobi morphism $\widehat{\mu} : L_M \to L_0$,
- there is a unique Jacobi structure $\{-,-\}_{M/\mathcal{G}}$ on $L_M/\mathcal{G} \to M/\mathcal{G}$ s.t. the quotient map $\widehat{q} : L_M \to L_M/\mathcal{G}$ is a Jacobi morphism covering $q : M \to M/\mathcal{G}$.

Proposition The following is a full contact dual pair, with connected fibers,

$$(M/\mathcal{G}, L_{M/\mathcal{G}}, \{-,-\}_{M/\mathcal{G}}) \xleftarrow{\widehat{q}} (M, \mathcal{H}_M) \xrightarrow{\widehat{\mu}} (\mathcal{G}_0^\mu, L_0, \{-,-\}_0).$$

This yields a new insight on the Contact Reduction [Zambon & Zhu]

- the *reduced orbit spaces* $\mu^{-1}(\mathcal{O})/\mathcal{G}$, as $\mathcal{O}$ varies among the coadjoint orbits in $\mathcal{G}_0^\mu$, are the characteristic leaves of $(M/\mathcal{G}, L_M/\mathcal{G}, \{-,-\}_{M/\mathcal{G}})$, and
- the *reduced transitive Jacobi structure* of $\mu^{-1}(\mathcal{O})/\mathcal{G}$ agree with the one inherited from $M/\mathcal{G}$.

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Thank you!