Orbifold equivalence for simple, unimodal and bimodal singularities

Ana Ros Camacho

Mathematisch Instituut, Universiteit Utrecht (Utrecht, The Netherlands)

XXVII IFWGP, September 6th, 2018

Universiteit Utrecht

Nederlandse Organisatie voor Wetenschappelijk Onderzoek

Motivation

What is "orbifold equivalence"?

- A particular kind of QFT: **Landau-Ginzburg models**.
- [Vafa-Warner,'88] A 2 dimensional, $(2,2)$-supersymmetric sigma model characterized by a **potential**.
- **Wanted**: an equivalence relation between two Landau-Ginzburg models described by different potentials.
- 'Orbifold'? Inspired by the study of orbifolds via defects in 2d QFTs.
- Way to go: via **matrix factorizations**.

Why?

- Interesting applications for the (so-called) Landau-Ginzburg/conformal field theory correspondence (AMA!).
- (Seems to be) Related to mirror symmetry, strange duality of singularities à l'Arnold, et al.

Introducing: matrix factorizations

Let $\Bbbk$ be a field $(=\mathbb{C})$, $S := \Bbbk[x_0, \ldots, x_n]$ polynomial ring. Assign $|x_i| \in \mathbb{Q}_{\geq 0}$ to each x_i. Consider $W \in S$.

Definition

- *W is called a* **potential** *if it has an isolated singularity at the origin. Equivalently:* $\dim_{\Bbbk}\left(\frac{S}{\langle \partial_{x_0} W, \ldots, \partial_{x_n} W\rangle}\right) < \infty$.
- *We say that a potential W is* **homogeneous of degree d** $\in \mathbb{Q}_{\geq 0}$ *if it satisfies* $W\left(\lambda^{|x_0|} x_0, \ldots, \lambda^{|x_n|} x_n\right) = \lambda^d W(x_0, \ldots, x_n)$ *for all* $\lambda \in \mathbb{C}^{\times}$.

From now on: potential= "homogeneous potential of degree 2". To any potential W we can associate a **central charge**, defined as: $c_W = \sum_{i=0}^{n} (1 - |x_i|)$.

Denote as $\mathcal{P}_{\Bbbk}$ the set of potentials with coefficients in $\Bbbk$ and any number of variables.

GOAL: define an equivalence relation in $\mathcal{P}_{\Bbbk}$!
HOW? **Matrix factorizations!**

Definition

*Given (S, W), a **matrix factorization** of W consists of a pair $\left(M, d^M\right)$ where:*

- *M free, $\mathbb{Z}_2$-graded $(= M_0 \oplus M_1)$ S-module, and*
- *$d^M\colon M \to M$ degree 1 $\left(= \begin{pmatrix} 0 & d_1^M \\ d_0^M & 0 \end{pmatrix}\right)$ S-linear endomorphism satisfying that*

$$d^M \circ d^M = W.\mathrm{id}_M$$

("twisted differential").

Example: let $S = \mathbb{C}[x]$, $W = x^d$. Possible matrix factorization:

$$\left(\mathbb{C}[x]^{\oplus 2}, \begin{pmatrix} 0 & x^m \\ x^{d-m} & 0 \end{pmatrix}\right), \text{ for } 0 < m < d.$$

Remark

We can use bimodules instead of left modules. Given $(S_1, W_1), (S_2, W_2)$ $\rightsquigarrow (S_1 \otimes_{\Bbbk} S_2, W_1 \otimes_{\Bbbk} 1_{S_2} - 1_{S_1} \otimes_{\Bbbk} W_2)$ (or simply: $W_1 - W_2$).

Using bimodules...

Definition

Let:

(S_1, W_1), (S_2, W_2), (S_3, W_3);

$\left(M, d^M\right)$ *matrix factorization of* $W_1 - W_2$, *and*

$\left(N, d^N\right)$ *matrix factorization of* $W_2 - W_3$.

The **tensor product matrix factorization** $\left(M \otimes_{S_2} N, d^{M \otimes N}\right)$ *is the matrix factorization of* $W_1 - W_3$ *with:*

- *base module over* $S_1 \otimes_{\mathbb{k}} S_3$,
- *twisted differential:* $d^{M \otimes N} = d^M \otimes \mathrm{id}_N + \mathrm{id}_M \otimes d^N$.

Construct a category: $\mathbf{hmf(W)}$

Objects: matrix factorizations of W,

Morphisms: given $\left(M, d^M\right), \left(N, d^N\right)$, $f \colon M \to N$ satisfying that $d^N \circ f = f \circ d^M$ (up to some homotopy relation).

Warning: categorical nonsense!

Theorem

- *[Carqueville-Runkel, '09]:* $\mathrm{hmf}\,(W \otimes 1 - 1 \otimes W)$ *is a monoidal category.*
- *[Carqueville-Murfet, '12]: every object in* $\mathrm{hmf}\,(W_1 - W_2)$ *has a left and a right adjoint and its evaluation and coevaluation morphisms can be described in a very explicit way.*

Remark

Results obtained from the study of Landau-Ginzburg models from a higher categorical perspective.

A particular example of a composition of these evaluation and coevaluation maps:

left and right quantum dimension (qdim_l, qdim_r resp.)

Orbifold equivalence

Definition/Proposition

Let $V(x_1, \ldots, x_n), W(y_1, \ldots, y_n) \in \mathcal{P}_{\Bbbk}$, $(M, d^M) \in \mathrm{Ob}(\mathrm{hmf}(W - V))$. *Assign to* (M, d^M) *two numbers,*

$$\mathrm{qdim}_l(M) = (-1)^{\binom{m+1}{2}} \mathrm{Res}\left[\frac{\mathrm{str}\left(\partial_{x_1} d^M \ldots \partial_{x_m} d^M \partial_{y_1} d^M \ldots \partial_{y_n} d^M\right) dy_1 \ldots dy_n}{\partial_{y_1} W, \ldots, \partial_{y_n} W}\right]$$

$$\mathrm{qdim}_r(M) = (-1)^{\binom{n+1}{2}} \mathrm{Res}\left[\frac{\mathrm{str}\left(\partial_{x_1} d^M \ldots \partial_{x_m} d^M \partial_{y_1} d^M \ldots \partial_{y_n} d^M\right) dx_1 \ldots dx_m}{\partial_{x_1} V, \ldots, \partial_{x_m} V}\right].$$

We say that V *and* W *are* **orbifold equivalent** *(notation:* $V \sim_{\mathrm{orb}} W$*) if there exists such an* (M, d^M) *for which* $\mathrm{qdim}_l, \mathrm{qdim}_r \neq 0$. *Orbifold equivalence is indeed an equivalence relation in* $\mathcal{P}_{\Bbbk}$.

Remark

- $\mathrm{qdim}_l, \mathrm{qdim}_r \in \Bbbk$.
- *Value of quantum dimensions are independent of the $\mathbb{Q}$-grading.*
- *If $V \sim_{\mathrm{orb}} W$, then $c_V = c_W$ and $m - n$ is even.*
- *Apparently not so easy to find...!*

Example: the quantum dimensions of

$$\left(\mathbb{C}[x,y]^{\oplus 2}, \begin{pmatrix} 0 & x-y \\ \frac{x^d - y^d}{x-y} & 0 \end{pmatrix} \right)$$

are...

$$\ldots \pm 1!$$

Examples of orbifold equivalence

[Arnold, 70s]: **classification of singularities**

Theorem (Carqueville-Runkel-RC,'13)

Consider the subset of $\mathcal{P}_{\mathbb{C}}$ which describes **simple singularities**. *They fall into an ADE classification and in 2 variables they look like:*

$$W_{A_{d-1}} = x^d + y^2, \qquad W_{D_{d+1}} = x^d + xy^2,$$
$$W_{E_6} = x^3 + y^4, \qquad W_{E_7} = x^3 + xy^3, \qquad W_{E_8} = x^3 + y^5$$

Then,

For d even, $\neq 12, 18, 30$, $W_{A_{d-1}} \sim_{orb} W_{D_{\frac{d}{2}+1}}$,

$W_{A_{11}} \sim_{orb} W_{D_7} \sim_{orb} W_{E_6}$,

$W_{A_{17}} \sim_{orb} W_{D_{10}} \sim_{orb} W_{E_7}$, *and*

$W_{A_{29}} \sim_{orb} W_{D_{16}} \sim_{orb} W_{E_8}$.

An example of a matrix factorization proving orbifold equivalence...

[Carqueville-RC-Runkel,'13]: take $S = \mathbb{C}[u, v, x, y]$ and consider the potential $W = W_{A_{11}} - W_{E_6} = u^{12} + y^2 - x^3 - y^4$.
A matrix factorization of this potential is $\left(\mathbb{C}[u, v, x, y]^{\oplus 4}, d^M\right)$, with

$$d_1^M = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}, \qquad d_0^M = \mathrm{Det}\left[d_1^M\right]\left(d_1^M\right)^{-1}$$

where

$$d_{11} = y^2 - v + \frac{1}{2}x(su)^2 + \frac{2t+1}{8}(su)^6$$

$$d_{12} = -x + y(su) + \frac{t+1}{4}(su)^4$$

$$d_{21} = x^2 + yx(su) + \frac{t}{4}x(su)^4 + \frac{2t+1}{4}y(su)^5 - \frac{9t+5}{48}(su)^8$$

$$d_{22} = y^2 + v + \frac{1}{2}x(su)^2 + \frac{2t+1}{8}(su)^6$$

(s and t can be any solution of $t^2 = \frac{1}{3}$, $s^{12} = -576(26t - 15)$).

Next on Arnold's classification: **unimodal singularities**. Focus on the exceptional ones.

Type	Potential (v1)	Potential (v2)	Potential (v3)	c_W
Q_{10}	$x^4+y^3+xz^2$	–	–	$\frac{13}{12}$
Q_{11}	$x^3y+y^3+xz^2$	–	–	$\frac{10}{9}$
Q_{12}	$x^3z+y^3+xz^2$	$x^5+y^3+xz^2$	–	$\frac{17}{15}$
S_{11}	$x^4+y^2z+xz^2$	–	–	$\frac{9}{8}$
S_{12}	$x^3y+y^2z+xz^2$	–	–	$\frac{15}{13}$
U_{12}	$x^4+y^3+z^3$	$x^4+y^3+z^2y$	$x^4+y^2z+z^2y$	$\frac{7}{6}$
Z_{11}	$x^5+xy^3+z^2$	–	–	$\frac{16}{15}$
Z_{12}	$yx^4+xy^3+z^2$	–	–	$\frac{12}{11}$
Z_{13}	$x^3z+xy^3+z^2$	$x^6+y^3x+z^2$	–	$\frac{10}{9}$
W_{12}	$x^5+y^2z+z^2$	$x^5+y^4+z^2$	–	$\frac{11}{10}$
W_{13}	$yx^4+y^2z+z^2$	$x^4y+y^4+z^2$	–	$\frac{9}{8}$
E_{12}	$x^7+y^3+z^2$	–	–	$\frac{22}{21}$
E_{13}	$y^3+yx^5+z^2$	–	–	$\frac{16}{15}$
E_{14}	$x^4z+y^3+z^2$	$x^8+y^3+z^2$	–	$\frac{13}{12}$

Question: which ones could be orbifold equivalent?

Next on Arnold's classification: **unimodal singularities**. Focus on the exceptional ones.

Type	Potential (v1)	Potential (v2)	Potential (v3)	c_W
Q_{10}	$x^4 + y^3 + xz^2$	–	–	$\frac{13}{12}$
E_{14}	$x^4 z + y^3 + z^2$	$x^8 + y^3 + z^2$	–	$\frac{13}{12}$

Next on Arnold's classification: **unimodal singularities**. Focus on the exceptional ones.

Type	Potential (v1)	Potential (v2)	Potential (v3)	c_W
Q_{11}	$x^3y + y^3 + xz^2$	–	–	$\frac{10}{9}$
Z_{13}	$x^3z + xy^3 + z^2$	$x^6 + y^3x + z^2$	–	$\frac{10}{9}$

Next on Arnold's classification: **unimodal singularities**. Focus on the exceptional ones.

Type	Potential (v1)	Potential (v2)	Potential (v3)	c_W
S_{11}	$x^4 + y^2 z + xz^2$	–	–	$\frac{9}{8}$
W_{13}	$yx^4 + y^2 z + z^2$	$x^4 y + y^4 + z^2$	–	$\frac{9}{8}$

Next on Arnold's classification: **unimodal singularities**. Focus on the exceptional ones.

Type	Potential (v1)	Potential (v2)	Potential (v3)	c_W
Z_{11}	$x^5 + xy^3 + z^2$	–	–	$\frac{16}{15}$
E_{13}	$y^3 + yx^5 + z^2$	–	–	$\frac{16}{15}$

Conjecture ('Arnold strange duality for exceptional unimodal singularities')

$Q_{10} \sim_{orb} E_{14}$, $Q_{11} \sim_{orb} Z_{13}$, $S_{11} \sim_{orb} W_{13}$ *and* $Z_{11} \sim_{orb} E_{13}$.

Proposition

1. *(Newton-RC,'15) Indeed* $Q_{10} \sim_{orb} E_{14}$. *Let* $\left(M_1, d^{M_1}\right)$ *(resp.* $\left(M_2, d^{M_2}\right)$*) be the matrix factorization proving orbifold equivalence betwen* Q_{10} *and* E_{14} *(v1) (resp. (v2)).* $\left(M_1, d^{M_1}\right)$ *(resp.* $\left(M_2, d^{M_2}\right)$*) depends on a set of parameters satisfying a certain set of conditions. The solutions of this system of equations are actually permuted by a Galois group isomorphic to* $D_4 \times C_2$ *(resp.* V_4*).*
2. *(Newton-RC,'16) Different potentials describing the same exceptional unimodal singularities are orbifold equivalent. The matrix factorizations proving orbifold equivalence (...) Galois group isomorphic to:*

Singularity	*Galois group*	*Singularity*	*Galois group*
E_{14}	D_8	W_{12}	C_2
Q_{12}	$C_2 \times C_5 \rtimes C_4$	W_{13}	D_8
U_{12} *(v2 vs v1/v3)*	$S_3 \times C_2$ / C_2	Z_{13}	$C_2 \times C_9 \rtimes C_6$

Remaining cases proved by Recknagel et al, '17 - but without Galois groups!

Next on Arnold's classification: **bimodal singularities**.

Conjecture ('Arnold strange duality for bimodal singularities')

$J_{3,0} \sim_{orb} Z_{13}$, $Q_{2,0} \sim_{orb} Z_{17}$, $E_{18} \sim_{orb} Q_{12}$, $E_{19} \sim_{orb} Z_{1,0}$, $Z_{19} \sim_{orb} E_{25}$, $Q_{17} \sim_{orb} Z_{2,0}$, $Q_{18} \sim_{orb} E_{30}$, $W_{17} \sim_{orb} S_{1,0}$, $S_{17} \sim_{orb} X_{2,0}$.

Proposition (RC '17, in preparation)

Indeed $Q_{18} \sim_{orb} E_{30}$ *and* $E_{18} \sim_{orb} Q_{12}$.

Sure but - very heavy computational problem. Can we simplify this search?

Result (Cornelissen-Kluck-RC, '18, work in progress)

- *Sage algorithm (interface in Julia) capable of deciding if two potentials are orbifold equivalent.*
- *If so, we can prove orbifold equivalence in finite time.*

Open questions

Few examples remaining involving bimodal singularities. More general instances of potentials?

Galois groups: missing ones?

Better, faster, stronger: improving the CKRC algorithm to speed up the search.

Applications to Landau-Ginzburg/conformal field theory correspondence?

Thank you very much for your attention! =)