Discrete geometry of polygons and Hamiltonian structures

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Discrete geometric realizations

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the Volterra model. We say that the projective tangential flow is a projective realization of the Volterra model.

2. If $V_n \in \mathbb{R}^2$ is a polygon in equicentro-affine plane ($\mathrm{SL}(2,\mathbb{R})$ acting linearly), with invariants

$$a_n = \det(V_n, V_{n+1}), \qquad \kappa_n = \frac{\det(V_n, V_{n+2})}{\det(V_{n+1}, V_{n+2})}$$

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then, whenever

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and $p_n = \dfrac{a_n}{a_{n+1}}$, $q_n = \kappa_n$, we have

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Authors: Bobenko, Doliwa& Santini, Fukujioka& Kurose, Hoffman, Mansfield, Marí Beffa, Inoguchi-Kajiwara & Matsuura, Wang, Suris.

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Then, curvature and torsion of the flow satisfy an equation equivalent to the Nonlinear Schrödinger equation (NLS). If

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We say the Vortex Filament flow is an Euclidean realization of NLS.Or NLS is the Vortex filament flow defined on the moduli space of Euclidean curves.

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Joint work with J. P. Wang, "Hamiltonian structures and integrable evolutions of twisted polygons in $\mathbb{RP}^n$", (2013), and A. Calini "Integrable evolutions of twisted polygons in centro-affine $\mathbb{R}^m$" (in progress).

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Let $M = G/H$ with G semisimple Lie group acting naturally on M. We say a polygon $\{u_n\} \in M^\infty$ $u_n \in M$ is twisted with period N if there exists $g \in G$ such that $u_{N+n} = g \cdot u_n$, for all n. The element g is called the monodromy of the polygon.

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How can we find coordinates for the moduli space of twisted polygons?

We define a right (resp. left) discrete moving frame associated to $\{u_n\}$ as an equivariant map

$$\rho : U \subset M^N \to G^N$$

wrt the *diagonal action* on M^N and the *right inverse* (resp. left) action on G^N. If $\rho(u) = (\rho_n)$, we say ρ_n is the moving frame at the vertex u_n.

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$\{K_n\}_{n=1}^N$ define (local) coordinates in the moduli space of polygons under the action of G.

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$$\begin{array}{rcl} \rho : (\mathbb{RP}^m)^N & \to & \mathrm{SL}(m+1, \mathbb{R})^N \\ \rho(\{u_n\}) & = & \{(V_{n+m}, \dots, V_{n+1}, V_n)\} \end{array}$$

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$$\rho_{n+1} = \rho_n \begin{pmatrix} k_n^m & 1 & 0 & \dots 0 & \\ k_n^{m-1} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \dots & \vdots \\ k_n^1 & 0 & \dots & 0 & 1 \\ (-1)^{m+1} & 0 & 0 & \dots & 0 \end{pmatrix}$$

where k_n^i are given by $V_{n+m+1} = k_n^m V_{n+m} + \dots k_n^1 V_{n+1} + (-1)^m V_n$ and $\{k_n^i\}$ generate all other invariants of the polygon.

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In general

Theorem

Assume $M = G/H$. The moduli space of non degenerate twisted polygons in M^N can be identified with an open subset of the quotient G^N/H^N, where H^N acts on G^N via the right gauge action

$$\begin{array}{ccc} H^N \times G^N & \to & G^N \\ ((h_n),(g_n)) & \to & (h_{n+1} g_n h_n^{-1}) \end{array}$$

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They were classified by Semenov-Tian-Shansky in "Dressing transformations and Poisson Group actions", (1985). We will describe the main bracket.

A Poisson structure on G^N

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$$R(\xi_+ + \xi_h + \xi_-) = \frac{1}{2}(\xi_+ - \xi_-).$$

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Associated to R there exists a 2-tensor r such that

$$r(\xi \wedge \eta) = \langle \xi, R(\eta)\rangle, \quad r(\xi \otimes \eta) = \langle \xi_+, \eta_- \rangle.$$

Define the twisted Poisson bracket in G^N to be given by

$$\{\mathcal{F},\mathcal{G}\}(L) = \sum_{s=1}^N r(\nabla_s\mathcal{F}\wedge\nabla_s\mathcal{G}) + \sum_{s=1}^N r(\nabla_s'\mathcal{F}\wedge\nabla_s'\mathcal{G})$$

$$-\sum_{s=1}^N r\left((\tau\otimes \mathrm{id})(\nabla_s'\mathcal{F}\otimes\nabla_s\mathcal{G})\right) + \sum_{s=1}^N r\left((\tau\otimes \mathrm{id})(\nabla_s'\mathcal{G}\otimes\nabla_s\mathcal{F})\right).$$

The twisted Poisson bracket defines a Hamiltonian structure for which gauge action is a Poisson map.

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Theorem

(MB 14) Assume $M = G/H$ and $\mathfrak{g}$ has two compatible gradations as above. The twisted Poisson structure defined on G^N, with r associated to the classical R-matrix, is locally reducible to the quotient G^N/H^N.

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Theorem

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The projective case

Example

(MB, Wang 13) In the projective case $G = \mathrm{PSL}(m+1)$ the gradations are:

1. $\mathfrak{g}_+ =$ strictly lower triangular matrices; $\mathfrak{h}_0 =$ diagonal matrices; $\mathfrak{g}_- =$ strictly upper triangular matrices.

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2. $$\mathfrak{g}_1 = \begin{pmatrix} 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 \\ * & \dots & * & 0 \end{pmatrix}, \mathfrak{g}_0 = \begin{pmatrix} * & \dots & * & 0 \\ \vdots & \vdots & \vdots & \vdots \\ * & \dots & * & 0 \\ 0 & \dots & 0 & * \end{pmatrix}, \mathfrak{g}_{-1} = \begin{pmatrix} 0 & \dots & 0 & * \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & * \\ 0 & \dots & 0 & 0 \end{pmatrix}$$

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If we define $\mathfrak{h} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, then $\mathbb{RP}^m = G/H$ is the standard description of the projective space as homogeneous space.
The two gradations are compatible and hence we have a reduced bracket.

In the case $n = 1$ the bracket is given by: let f, g be functions of k_n

$$\{f, g\}(k_n) = \sum_n \frac{df}{dk_n} \left(\tau^{-1} - \tau + k_n(\tau - 1)(\tau + 1)^{-1} k_n\right) \frac{dg}{dk_n}$$

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Theorem

(MB, Wang 13) The evolution in $\mathbb{RP}^m$ whose unique lift to $\mathbb{R}^{m+1}$ is

$$(V_n)_t = \frac{1}{k_{n-1}^1}\left(V_{n+m} + k_n^m V_{n+m-1} + \cdots + k_n^2 V_{n+1}\right) + \alpha_n V_n$$

α_n uniquely determined by preservation of $\det(V_{n+m}, \ldots, V_n) = 1$, induces on k_n^i (bi-Hamiltonian?) and completely integrable discretizations of W_m algebras, (generalizations of the Boussinesq lattice), for any m.

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The equations are given by

$$\begin{cases} k^i_t = \frac{k^{i+1}_1}{k^1_1} - \frac{k^{i+1}}{k^1_{-i}}, & i = 1, 2, \cdots, m-1 \\ k^m_t = \frac{1}{k^1_1} - \frac{1}{k^1_{-N}} \end{cases}$$

Under the *Miura* transformation $u^1 = \frac{1}{k^1 k^1_1 \cdots k^1_n}$, $\quad u^i = \frac{k^i_{i-1}}{k^1 k^1_1 \cdots k^1_{i-1}}$, $i = 2, \cdots, N+1$, they become

$$\begin{cases} u^1_t = -u^1(u^2_N - u^2_{-1}) \\ u^i_t = u^{i+1} - u^{i+1}_{-1} - u^i(u^2_{i-1} - u^2_{-1}), & i = 2, 3 \cdots, m-1 \\ u^m_t = u^1 - u^1_{-1} - u^m(u^2_{N-1} - u^2_{-1}) \end{cases}$$

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(MB, Wang 13) The right bracket for the parabolic r tensor

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They were proved to be compatible in recent work with Calini. The study of the compatibility of the Poisson pair is based on lifting the two Hamiltonian structures to two pre-symplectic structures in the space of projective polygons, and study the properties at that level.

Theorem

(Calini, MB, 2017) Given a Hamiltonian f on the moduli space, there exists moduli coordinates $\mathbf{a} = (a_n)$ and a vector field Y^f such that if $(u_n)_t = Y^f_n$, then $(a_n)_t$ is f-Hamiltonian wrt to the main reduced bracket $\{,\}_1$.

Furthermore, there exists pre-symplectic forms *ω_1 and ω_2 on the space of polygons such that*

$$\omega_1(Y^f, Y^g)(u) = \{f, g\}_1(\mathbf{a}), \qquad \omega_2(Y^f, Y^g)(u) = \{f, g\}_2(\mathbf{a}).$$

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$$\omega_1(X, Y)(u) = \frac{1}{2} \sum_n \sum_{r=1}^{m-1} \{\det(J_n(Y), V_{n+1}, \dots, X_{n+r}, \dots, V_{n+m-1})$$

$$-\det(J_n(X), V_{n+1}, \dots, Y_{n+r}, \dots, V_{n+m-1})$$

$$+\det(V_n, \dots, Y_{n+r-1}, \dots, X_{n+m-1}) - \det(V_n, \dots, X_{n+r-1}, \dots, Y_{n+m-1})\}$$

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Theorem

(Calini, MB, 18) There exist two integrable hierarchies associated to each of two vector fields generating the kernel of ω_2*. Restricted to the moduli space,* ω_1 *is symplectic and one can generate a recursion operator.*

GRACIAS!

THANKS!