SOLVABILITY IMPLIES INTEGRABILITY

Janusz Grabowski
(Polish Academy of Sciences)

XXVII International Fall Workshop on Geometry and Physics
Sevilla, 3-7 September, 2018

Contents

- Integrability by quadratures
- Rectification
- Integrability by quadratures - examples
- Solvable Lie algebras
- Lie's Theorem
- Main result
- Abelian Lie ideals
- Rectification of Abelian algebras of vector fields
- Sketch of the proof and example

The talk is based on a joint work with J. F. Cariñena and F. Falceto:

- Solvability of a Lie algebra of vector fields implies their integrability by quadratures, *J. Phys. A* **49** (2016), no. 42, 425202, 13 pp.

Contents

- Integrability by quadratures
- Rectification
- Integrability by quadratures - examples
- Solvable Lie algebras
- Lie's Theorem
- Main result
- Abelian Lie ideals
- Rectification of Abelian algebras of vector fields
- Sketch of the proof and example

The talk is based on a joint work with J. F. Cariñena and F. Falceto:

- Solvability of a Lie algebra of vector fields implies their integrability by quadratures, *J. Phys. A* **49** (2016), no. 42, 425202, 13 pp.

Contents

- Integrability by quadratures
- Rectification
- Integrability by quadratures - examples
- Solvable Lie algebras
- Lie's Theorem
- Main result
- Abelian Lie ideals
- Rectification of Abelian algebras of vector fields
- Sketch of the proof and example

The talk is based on a joint work with J. F. Cariñena and F. Falceto:

- Solvability of a Lie algebra of vector fields implies their integrability by quadratures, *J. Phys. A* **49** (2016), no. 42, 425202, 13 pp.

Contents

The talk is based on a joint work with J. F. Cariñena and F. Falceto:

- Solvability of a Lie algebra of vector fields implies their integrability by quadratures, *J. Phys. A* **49** (2016), no. 42, 425202, 13 pp.

Contents

The talk is based on a joint work with J. F. Cariñena and F. Falceto:

- Solvability of a Lie algebra of vector fields implies their integrability by quadratures, *J. Phys. A* **49** (2016), no. 42, 425202, 13 pp.

Contents

The talk is based on a joint work with J. F. Cariñena and F. Falceto:

- Solvability of a Lie algebra of vector fields implies their integrability by quadratures, *J. Phys. A* **49** (2016), no. 42, 425202, 13 pp.

Contents

The talk is based on a joint work with J. F. Cariñena and F. Falceto:

- Solvability of a Lie algebra of vector fields implies their integrability by quadratures, *J. Phys. A* **49** (2016), no. 42, 425202, 13 pp.

Contents

- Integrability by quadratures
- Rectification
- Integrability by quadratures - examples
- Solvable Lie algebras
- Lie's Theorem
- Main result
- Abelian Lie ideals
- Rectification of Abelian algebras of vector fields
- Sketch of the proof and example

The talk is based on a joint work with J. F. Cariñena and F. Falceto:

- Solvability of a Lie algebra of vector fields implies their integrability by quadratures, *J. Phys. A* **49** (2016), no. 42, 425202, 13 pp.

Contents

The talk is based on a joint work with J. F. Cariñena and F. Falceto:

- Solvability of a Lie algebra of vector fields implies their integrability by quadratures, *J. Phys. A* **49** (2016), no. 42, 425202, 13 pp.

Contents

- Integrability by quadratures
- Rectification
- Integrability by quadratures - examples
- Solvable Lie algebras
- Lie's Theorem
- Main result
- Abelian Lie ideals
- Rectification of Abelian algebras of vector fields
- Sketch of the proof and example

The talk is based on a joint work with J. F. Cariñena and F. Falceto:

- Solvability of a Lie algebra of vector fields implies their integrability by quadratures, *J. Phys. A* **49** (2016), no. 42, 425202, 13 pp.

Contents

- Integrability by quadratures
- Rectification
- Integrability by quadratures - examples
- Solvable Lie algebras
- Lie's Theorem
- Main result
- Abelian Lie ideals
- Rectification of Abelian algebras of vector fields
- Sketch of the proof and example

The talk is based on a joint work with J. F. Cariñena and F. Falceto:

- Solvability of a Lie algebra of vector fields implies their integrability by quadratures, *J. Phys. A* **49** (2016), no. 42, 425202, 13 pp.

Contents

- Integrability by quadratures
- Rectification
- Integrability by quadratures - examples
- Solvable Lie algebras
- Lie's Theorem
- Main result
- Abelian Lie ideals
- Rectification of Abelian algebras of vector fields
- Sketch of the proof and example

The talk is based on a joint work with J. F. Cariñena and F. Falceto:

- Solvability of a Lie algebra of vector fields implies their integrability by quadratures, *J. Phys. A* **49** (2016), no. 42, 425202, 13 pp.

Integrability by quadratures

- An autonomous system of differential equations,

$$\dot{x}^i = f^i(x^1, \ldots, x^n)\,, \qquad i = 1, \ldots, n, \tag{1}$$

is geometrically interpreted in terms of a vector field Γ in a n-dimensional manifold M with a local expression

$$\Gamma = \sum_{i=1}^{n} f^i(x^1, \ldots, x^n)\partial_{x^i}\,.$$

- The integral curves of Γ are the solutions of (1). Integrating the system amounts to determine its general solution.
- In particular, we speak about the integrability by quadratures if you can determine the solutions by means of a finite number of elementary functions (in particular, algebraic operations) and integrations of known functions. Historically, this is the first concept of integrability.

Integrability by quadratures

- An autonomous system of differential equations,

$$\dot{x}^i = f^i(x^1, \ldots, x^n)\,, \qquad i = 1, \ldots, n, \tag{1}$$

is geometrically interpreted in terms of a vector field Γ in a n-dimensional manifold M with a local expression

$$\Gamma = \sum_{i=1}^{n} f^i(x^1, \ldots, x^n)\partial_{x^i}\,.$$

- The integral curves of Γ are the solutions of (1). Integrating the system amounts to determine its general solution.
- In particular, we speak about the integrability by quadratures if you can determine the solutions by means of a finite number of elementary functions (in particular, algebraic operations) and integrations of known functions. Historically, this is the first concept of integrability.

Integrability by quadratures

- An autonomous system of differential equations,

$$\dot{x}^i = f^i(x^1, \ldots, x^n) \,, \qquad i = 1, \ldots, n, \tag{1}$$

  is geometrically interpreted in terms of a vector field Γ in a n-dimensional manifold M with a local expression

$$\Gamma = \sum_{i=1}^{n} f^i(x^1, \ldots, x^n) \partial_{x^i} \,.$$

- The integral curves of Γ are the solutions of (1). Integrating the system amounts to determine its general solution.
- In particular, we speak about the integrability by quadratures if you can determine the solutions by means of a finite number of elementary functions (in particular, algebraic operations) and integrations of known functions. Historically, this is the first concept of integrability.

Integrability by quadratures

- An autonomous system of differential equations,

$$\dot{x}^i = f^i(x^1, \ldots, x^n) \ , \qquad i = 1, \ldots, n, \tag{1}$$

  is geometrically interpreted in terms of a vector field Γ in a n-dimensional manifold M with a local expression

$$\Gamma = \sum_{i=1}^{n} f^i(x^1, \ldots, x^n)\partial_{x^i} \ .$$

- The integral curves of Γ are the solutions of (1). Integrating the system amounts to determine its general solution.
- In particular, we speak about the integrability by quadratures if you can determine the solutions by means of a finite number of elementary functions (in particular, algebraic operations) and integrations of known functions. Historically, this is the first concept of integrability.

Integrability by quadratures

- An autonomous system of differential equations,

$$\dot{x}^i = f^i(x^1, \ldots, x^n) \ , \qquad i = 1, \ldots, n, \tag{1}$$

  is geometrically interpreted in terms of a vector field Γ in a n-dimensional manifold M with a local expression

$$\Gamma = \sum_{i=1}^{n} f^i(x^1, \ldots, x^n)\partial_{x^i} \, .$$

- The integral curves of Γ are the solutions of (1). Integrating the system amounts to determine its general solution.
- In particular, we speak about the integrability by quadratures if you can determine the solutions by means of a finite number of elementary functions (in particular, algebraic operations) and integrations of known functions. Historically, this is the first concept of integrability.

Integrability by quadratures

- An autonomous system of differential equations,

$$\dot{x}^i = f^i(x^1, \ldots, x^n)\ , \qquad i = 1, \ldots, n, \tag{1}$$

  is geometrically interpreted in terms of a vector field Γ in a n-dimensional manifold M with a local expression

$$\Gamma = \sum_{i=1}^{n} f^i(x^1, \ldots, x^n)\partial_{x^i}\ .$$

- The integral curves of Γ are the solutions of (1). Integrating the system amounts to determine its general solution.
- In particular, we speak about the integrability by quadratures if you can determine the solutions by means of a finite number of elementary functions (in particular, algebraic operations) and integrations of known functions. Historically, this is the first concept of integrability.

Integrability by quadratures

- This goes back to 1885 treatise of Maximovič, motivated by the well-known explicit solution formula for the linear first-order equation

$$y' + py = q\,.$$

- Despite Maximovičs professed intention to lay the foundation of an entirely new theory of an importance comparable to that of the theory of general algebraic equations, his work does not seem to have found a wider audience.
- An apparent outward reason for this is its practical inaccessibility as a monograph printed at the Imperial University at Kazan', unavailable even at the larger American and Western European libraries. Moreover, Maximovičs statements are often obscure and open to interpretation. Nevertheless, his work seems to contain some original, useful and justiffable ideas which deserve to be brought to light.

Integrability by quadratures

- This goes back to 1885 treatise of Maximovič, motivated by the well-known explicit solution formula for the linear first-order equation

$$y' + py = q\,.$$

- Despite Maximovičs professed intention to lay the foundation of an entirely new theory of an importance comparable to that of the theory of general algebraic equations, his work does not seem to have found a wider audience.
- An apparent outward reason for this is its practical inaccessibility as a monograph printed at the Imperial University at Kazan', unavailable even at the larger American and Western European libraries. Moreover, Maximovičs statements are often obscure and open to interpretation. Nevertheless, his work seems to contain some original, useful and justiffable ideas which deserve to be brought to light.

Integrability by quadratures

- This goes back to 1885 treatise of Maximovič, motivated by the well-known explicit solution formula for the linear first-order equation

$$y' + py = q\,.$$

- Despite Maximovičs professed intention to lay the foundation of an entirely new theory of an importance comparable to that of the theory of general algebraic equations, his work does not seem to have found a wider audience.
- An apparent outward reason for this is its practical inaccessibility as a monograph printed at the Imperial University at Kazan', unavailable even at the larger American and Western European libraries. Moreover, Maximovičs statements are often obscure and open to interpretation. Nevertheless, his work seems to contain some original, useful and justiffable ideas which deserve to be brought to light.

Integrability by quadratures

- This goes back to 1885 treatise of Maximovič, motivated by the well-known explicit solution formula for the linear first-order equation

$$y' + py = q\,.$$

- Despite Maximovičs professed intention to lay the foundation of an entirely new theory of an importance comparable to that of the theory of general algebraic equations, his work does not seem to have found a wider audience.
- An apparent outward reason for this is its practical inaccessibility as a monograph printed at the Imperial University at Kazan', unavailable even at the larger American and Western European libraries. Moreover, Maximovičs statements are often obscure and open to interpretation. Nevertheless, his work seems to contain some original, useful and justiffable ideas which deserve to be brought to light.

Integrability by quadratures

- This goes back to 1885 treatise of Maximovič, motivated by the well-known explicit solution formula for the linear first-order equation

$$y' + py = q\,.$$

- Despite Maximovičs professed intention to lay the foundation of an entirely new theory of an importance comparable to that of the theory of general algebraic equations, his work does not seem to have found a wider audience.
- An apparent outward reason for this is its practical inaccessibility as a monograph printed at the Imperial University at Kazan', unavailable even at the larger American and Western European libraries. Moreover, Maximovičs statements are often obscure and open to interpretation. Nevertheless, his work seems to contain some original, useful and justiffable ideas which deserve to be brought to light.

Integrability by quadratures

- In the first part of his work Maximovič claims to show that a symbolic first-order ordinary differential equation can be integrated by quadratures if and only if it arises from the linear first-order equation

$$y' + py = q\,.$$

by means of a transformation of the unknown variable y.
- In the second part he proceeds to find criteria for a given equation to have this property, and concludes, among other things, that the linear second-order equation (which is intimately connected with the non-linear first-order Riccati equation) cannot be integrated by quadratures in general.

Integrability by quadratures

- In the first part of his work Maximovič claims to show that a symbolic first-order ordinary differential equation can be integrated by quadratures if and only if it arises from the linear first-order equation

$$y' + py = q\,.$$

  by means of a transformation of the unknown variable y.
- In the second part he proceeds to find criteria for a given equation to have this property, and concludes, among other things, that the linear second-order equation (which is intimately connected with the non-linear first-order Riccati equation) cannot be integrated by quadratures in general.

Integrability by quadratures

- In the first part of his work Maximovič claims to show that a symbolic first-order ordinary differential equation can be integrated by quadratures if and only if it arises from the linear first-order equation

$$y' + py = q\,.$$

by means of a transformation of the unknown variable y.

- In the second part he proceeds to find criteria for a given equation to have this property, and concludes, among other things, that the linear second-order equation (which is intimately connected with the non-linear first-order Riccati equation) cannot be integrated by quadratures in general.

Rectification

- Note, however, that the concept of integrability by quadratures is computational and not geometric, as it depends on coordinates in which we work.
- The result of the straightening out (rectification) Theorem asserts the existence of coordinates $(y^1, \ldots, y^n)$ in a neighbourhood of a point where Γ is different from zero such that

$$\Gamma = \partial_{y^n}\,.$$

- The new coordinates $y^1, \ldots, y^{n-1}$, are constants of motion and therefore we cannot find easily such coordinates in a general case.
- It is clear that the if we use such rectifying coordinates for Γ the integration is immediate, the solution being

$$y^k(t) = y_0^k, \quad k = 1, \ldots, n-1, \qquad y^n(t) = y_0^n + t.$$

- This proves that the concept of integrability by quadratures depends on the choice of initial coordinates, because using these rectifying coordinates the system is always integrable by quadratures.

Rectification

- Note, however, that the concept of integrability by quadratures is computational and not geometric, as it depends on coordinates in which we work.
- The result of the straightening out (rectification) Theorem asserts the existence of coordinates $(y^1, \ldots, y^n)$ in a neighbourhood of a point where Γ is different from zero such that

$$\Gamma = \partial_{y^n}.$$

- The new coordinates $y^1, \ldots, y^{n-1}$, are constants of motion and therefore we cannot find easily such coordinates in a general case.
- It is clear that the if we use such rectifying coordinates for Γ the integration is immediate, the solution being

$$y^k(t) = y_0^k, \quad k = 1, \ldots, n-1, \qquad y^n(t) = y_0^n + t.$$

- This proves that the concept of integrability by quadratures depends on the choice of initial coordinates, because using these rectifying coordinates the system is always integrable by quadratures.

Rectification

- Note, however, that the concept of integrability by quadratures is computational and not geometric, as it depends on coordinates in which we work.
- The result of the straightening out (rectification) Theorem asserts the existence of coordinates $(y^1, \ldots, y^n)$ in a neighbourhood of a point where Γ is different from zero such that

$$\Gamma = \partial_{y^n}\,.$$

- The new coordinates $y^1, \ldots, y^{n-1}$, are constants of motion and therefore we cannot find easily such coordinates in a general case.
- It is clear that the if we use such rectifying coordinates for Γ the integration is immediate, the solution being

$$y^k(t) = y^k_0, \quad k = 1, \ldots, n-1, \qquad y^n(t) = y^n_0 + t.$$

- This proves that the concept of integrability by quadratures depends on the choice of initial coordinates, because using these rectifying coordinates the system is always integrable by quadratures.

Rectification

- Note, however, that the concept of integrability by quadratures is computational and not geometric, as it depends on coordinates in which we work.
- The result of the straightening out (rectification) Theorem asserts the existence of coordinates $(y^1, \ldots, y^n)$ in a neighbourhood of a point where Γ is different from zero such that

$$\Gamma = \partial_{y^n} .$$

- The new coordinates $y^1, \ldots, y^{n-1}$, are constants of motion and therefore we cannot find easily such coordinates in a general case.
- It is clear that the if we use such rectifying coordinates for Γ the integration is immediate, the solution being

$$y^k(t) = y_0^k, \quad k = 1, \ldots, n-1, \qquad y^n(t) = y_0^n + t.$$

- This proves that the concept of integrability by quadratures depends on the choice of initial coordinates, because using these rectifying coordinates the system is always integrable by quadratures.

Rectification

- Note, however, that the concept of integrability by quadratures is computational and not geometric, as it depends on coordinates in which we work.
- The result of the straightening out (rectification) Theorem asserts the existence of coordinates $(y^1, \ldots, y^n)$ in a neighbourhood of a point where Γ is different from zero such that

$$\Gamma = \partial_{y^n} .$$

- The new coordinates $y^1, \ldots, y^{n-1}$, are constants of motion and therefore we cannot find easily such coordinates in a general case.
- It is clear that the if we use such rectifying coordinates for Γ the integration is immediate, the solution being

$$y^k(t) = y_0^k, \quad k = 1, \ldots, n-1, \qquad y^n(t) = y_0^n + t.$$

- This proves that the concept of integrability by quadratures depends on the choice of initial coordinates, because using these rectifying coordinates the system is always integrable by quadratures.

Rectification

- Note, however, that the concept of integrability by quadratures is computational and not geometric, as it depends on coordinates in which we work.
- The result of the straightening out (rectification) Theorem asserts the existence of coordinates $(y^1, \dots, y^n)$ in a neighbourhood of a point where Γ is different from zero such that

$$\Gamma = \partial_{y^n} .$$

- The new coordinates $y^1, \dots, y^{n-1}$, are constants of motion and therefore we cannot find easily such coordinates in a general case.
- It is clear that the if we use such rectifying coordinates for Γ the integration is immediate, the solution being

$$y^k(t) = y_0^k, \quad k = 1, \dots, n-1, \qquad y^n(t) = y_0^n + t.$$

- This proves that the concept of integrability by quadratures depends on the choice of initial coordinates, because using these rectifying coordinates the system is always integrable by quadratures.

Integrability by quadratures - examples

- Consider first the non-autonomous inhomogeneous linear differential equation in dimension one,

$$\dot{x} = c_1(t)\, x + c_0(t)\,,$$

which is well known to be integrable in terms of two quadratures:

$$x(t) = \exp\left(\int_0^t c_1(t')\, dt'\right)\left[x_0 + \int_0^t \exp\left(-\int_0^{t'} c_1(t'')\, dt''\right) c_0(t')\, dt'\right].$$

- Another example is given by the nonautonomous system of differential equations

$$\dot{x}^i = \sum_{j=1}^n H^i{}_j\, x^j + b^i(t), \quad i = 1, \ldots, n,$$

where $H^i{}_j$ are real numbers. Then, the solution starting from the point $\boldsymbol{x}_0$ is given by

$$\boldsymbol{x}(t) = \exp(Ht)\left[\boldsymbol{x}_0 + \int_0^t \exp(-Ht')\, \boldsymbol{b}(t')dt'\right].$$

Integrability by quadratures - examples

- Consider first the non-autonomous inhomogeneous linear differential equation in dimension one,

$$\dot{x} = c_1(t)\, x + c_0(t)\,,$$

which is well known to be integrable in terms of two quadratures:

$$x(t) = \exp\left(\int_0^t c_1(t')\, dt'\right)\left[x_0 + \int_0^t \exp\left(-\int_0^{t'} c_1(t'')\, dt''\right) c_0(t')\, dt'\right].$$

- Another example is given by the nonautonomous system of differential equations

$$\dot{x}^i = \sum_{j=1}^{n} H^i{}_j\, x^j + b^i(t), \quad i = 1, \ldots, n,$$

where $H^i{}_j$ are real numbers. Then, the solution starting from the point $\mathbf{x}_0$ is given by

$$\mathbf{x}(t) = \exp(Ht)\left[\mathbf{x}_0 + \int_0^t \exp(-Ht')\, \mathbf{b}(t')dt'\right].$$

Integrability by quadratures - examples

- Consider first the non-autonomous inhomogeneous linear differential equation in dimension one,

$$\dot{x} = c_1(t)\, x + c_0(t)\,,$$

which is well known to be integrable in terms of two quadratures:

$$x(t) = \exp\left(\int_0^t c_1(t')\, dt'\right)\left[x_0 + \int_0^t \exp\left(-\int_0^{t'} c_1(t'')\, dt''\right) c_0(t')\, dt'\right].$$

- Another example is given by the nonautonomous system of differential equations

$$\dot{x}^i = \sum_{j=1}^{n} H^i{}_j\, x^j + b^i(t), \quad i = 1, \dots, n,$$

where $H^i{}_j$ are real numbers. Then, the solution starting from the point $\mathbf{x}_0$ is given by

$$\mathbf{x}(t) = \exp(Ht)\left[\mathbf{x}_0 + \int_0^t \exp(-Ht')\, \mathbf{b}(t') dt'\right].$$

Integrability by quadratures - examples

- Consider first the non-autonomous inhomogeneous linear differential equation in dimension one,

$$\dot{x} = c_1(t)\, x + c_0(t)\,,$$

which is well known to be integrable in terms of two quadratures:

$$x(t) = \exp\left(\int_0^t c_1(t')\, dt'\right) \left[x_0 + \int_0^t \exp\left(-\int_0^{t'} c_1(t'')\, dt''\right) c_0(t')\, dt'\right].$$

- Another example is given by the nonautonomous system of differential equations

$$\dot{x}^i = \sum_{j=1}^n H^i{}_j\, x^j + b^i(t), \quad i = 1, \dots, n,$$

where $H^i{}_j$ are real numbers. Then, the solution starting from the point $\boldsymbol{x}_0$ is given by

$$\boldsymbol{x}(t) = \exp(Ht)\left[\boldsymbol{x}_0 + \int_0^t \exp(-Ht')\; \boldsymbol{b}(t')dt'\right].$$

Integrability by quadratures - examples

- Consider first the non-autonomous inhomogeneous linear differential equation in dimension one,

$$\dot{x} = c_1(t)\,x + c_0(t)\,,$$

which is well known to be integrable in terms of two quadratures:

$$x(t) = \exp\left(\int_0^t c_1(t')\,dt'\right)\left[x_0 + \int_0^t \exp\left(-\int_0^{t'} c_1(t'')\,dt''\right) c_0(t')\,dt'\right].$$

- Another example is given by the nonautonomous system of differential equations

$$\dot{x}^i = \sum_{j=1}^n H^i{}_j\,x^j + b^i(t), \quad i = 1, \ldots, n,$$

where $H^i{}_j$ are real numbers. Then, the solution starting from the point $\boldsymbol{x}_0$ is given by

$$\boldsymbol{x}(t) = \exp\left(Ht\right)\left[\boldsymbol{x}_0 + \int_0^t \exp\left(-Ht'\right)\,\boldsymbol{b}(t')dt'\right].$$

Solvable Lie algebras

- Recall that the derived algebra of a Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$ is the subalgebra $\mathfrak{g}^1$ of $\mathfrak{g}$, defined by $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}]$, while the derived series is the sequence of Lie subalgebras defined by $\mathfrak{g}^0 = \mathfrak{g}$ and

$$\mathfrak{g}^{k+1} = [\mathfrak{g}^k, \mathfrak{g}^k], \quad k \in \mathbb{N}.$$

- Such a sequence satisfies $\mathfrak{g}^{k+1} \subset \mathfrak{g}^k$, and the Lie algebra $\mathfrak{g}$ is said to be solvable if the derived series eventually arrives at the zero subalgebra, i.e. there exists the smallest natural number m such that $\mathfrak{g}^{m+1} = \{0\}$ or, in other words, $\mathfrak{g}^m$ is Abelian.
- Example. The Lie algebra of upper-triangular matrices is solvable.

Solvable Lie algebras

- Recall that the derived algebra of a Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$ is the subalgebra $\mathfrak{g}^1$ of $\mathfrak{g}$, defined by $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}]$, while the derived series is the sequence of Lie subalgebras defined by $\mathfrak{g}^0 = \mathfrak{g}$ and

$$\mathfrak{g}^{k+1} = [\mathfrak{g}^k, \mathfrak{g}^k], \quad k \in \mathbb{N}\,.$$

- Such a sequence satisfies $\mathfrak{g}^{k+1} \subset \mathfrak{g}^k$, and the Lie algebra $\mathfrak{g}$ is said to be solvable if the derived series eventually arrives at the zero subalgebra, i.e. there exists the smallest natural number m such that $\mathfrak{g}^{m+1} = \{0\}$ or, in other words, $\mathfrak{g}^m$ is Abelian.
- Example. The Lie algebra of upper-triangular matrices is solvable.

$$\begin{pmatrix} * & & & * \\ & * & & \\ & & \ddots & \\ 0 & & & * \end{pmatrix}$$

Solvable Lie algebras

- Recall that the derived algebra of a Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$ is the subalgebra $\mathfrak{g}^1$ of $\mathfrak{g}$, defined by $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}]$, while the derived series is the sequence of Lie subalgebras defined by $\mathfrak{g}^0 = \mathfrak{g}$ and

$$\mathfrak{g}^{k+1} = [\mathfrak{g}^k, \mathfrak{g}^k], \quad k \in \mathbb{N}\,.$$

- Such a sequence satisfies $\mathfrak{g}^{k+1} \subset \mathfrak{g}^k$, and the Lie algebra $\mathfrak{g}$ is said to be solvable if the derived series eventually arrives at the zero subalgebra, i.e. there exists the smallest natural number m such that $\mathfrak{g}^{m+1} = \{0\}$ or, in other words, $\mathfrak{g}^m$ is Abelian.
- Example. The Lie algebra of upper-triangular matrices is solvable.

$$\begin{pmatrix} * & & & * \\ & \ddots & & \\ 0 & & \ddots & \\ & & & * \end{pmatrix}$$

Solvable Lie algebras

- Recall that the derived algebra of a Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$ is the subalgebra $\mathfrak{g}^1$ of $\mathfrak{g}$, defined by $\mathfrak{g}^1 = [\mathfrak{g},\mathfrak{g}]$, while the derived series is the sequence of Lie subalgebras defined by $\mathfrak{g}^0 = \mathfrak{g}$ and

$$\mathfrak{g}^{k+1} = [\mathfrak{g}^k, \mathfrak{g}^k], \quad k \in \mathbb{N}.$$

- Such a sequence satisfies $\mathfrak{g}^{k+1} \subset \mathfrak{g}^k$, and the Lie algebra $\mathfrak{g}$ is said to be solvable if the derived series eventually arrives at the zero subalgebra, i.e. there exists the smallest natural number m such that $\mathfrak{g}^{m+1} = \{0\}$ or, in other words, $\mathfrak{g}^m$ is Abelian.
- Example. The Lie algebra of upper-triangular matrices is solvable.

Solvable Lie algebras

- Recall that the derived algebra of a Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$ is the subalgebra $\mathfrak{g}^1$ of $\mathfrak{g}$, defined by $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}]$, while the derived series is the sequence of Lie subalgebras defined by $\mathfrak{g}^0 = \mathfrak{g}$ and

$$\mathfrak{g}^{k+1} = [\mathfrak{g}^k, \mathfrak{g}^k], \quad k \in \mathbb{N}\,.$$

- Such a sequence satisfies $\mathfrak{g}^{k+1} \subset \mathfrak{g}^k$, and the Lie algebra $\mathfrak{g}$ is said to be solvable if the derived series eventually arrives at the zero subalgebra, i.e. there exists the smallest natural number m such that $\mathfrak{g}^{m+1} = \{0\}$ or, in other words, $\mathfrak{g}^m$ is Abelian.
- Example. The Lie algebra of upper-triangular matrices is solvable.

Solvable Lie algebras

- Recall that the derived algebra of a Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$ is the subalgebra $\mathfrak{g}^1$ of $\mathfrak{g}$, defined by $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}]$, while the derived series is the sequence of Lie subalgebras defined by $\mathfrak{g}^0 = \mathfrak{g}$ and

$$\mathfrak{g}^{k+1} = [\mathfrak{g}^k, \mathfrak{g}^k], \quad k \in \mathbb{N}\,.$$

- Such a sequence satisfies $\mathfrak{g}^{k+1} \subset \mathfrak{g}^k$, and the Lie algebra $\mathfrak{g}$ is said to be solvable if the derived series eventually arrives at the zero subalgebra, i.e. there exists the smallest natural number m such that $\mathfrak{g}^{m+1} = \{0\}$ or, in other words, $\mathfrak{g}^m$ is Abelian.
- Example. The Lie algebra of upper-triangular matrices is solvable.

$$\begin{pmatrix} * & & & * \\ & \ddots & & \\ & & \ddots & \\ 0 & & & * \end{pmatrix}$$

Solvable Lie algebras

- Recall that the derived algebra of a Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$ is the subalgebra $\mathfrak{g}^1$ of $\mathfrak{g}$, defined by $\mathfrak{g}^1 = [\mathfrak{g},\mathfrak{g}]$, while the derived series is the sequence of Lie subalgebras defined by $\mathfrak{g}^0 = \mathfrak{g}$ and

$$\mathfrak{g}^{k+1} = [\mathfrak{g}^k, \mathfrak{g}^k], \quad k \in \mathbb{N}\,.$$

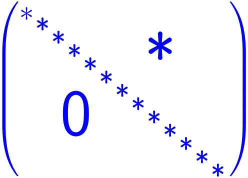
- Such a sequence satisfies $\mathfrak{g}^{k+1} \subset \mathfrak{g}^k$, and the Lie algebra $\mathfrak{g}$ is said to be solvable if the derived series eventually arrives at the zero subalgebra, i.e. there exists the smallest natural number m such that $\mathfrak{g}^{m+1} = \{0\}$ or, in other words, $\mathfrak{g}^m$ is Abelian.
- Example. The Lie algebra of upper-triangular matrices is solvable.

Solvable Lie algebras

- Recall that the derived algebra of a Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$ is the subalgebra $\mathfrak{g}^1$ of $\mathfrak{g}$, defined by $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}]$, while the derived series is the sequence of Lie subalgebras defined by $\mathfrak{g}^0 = \mathfrak{g}$ and

$$\mathfrak{g}^{k+1} = [\mathfrak{g}^k, \mathfrak{g}^k], \quad k \in \mathbb{N}\,.$$

- Such a sequence satisfies $\mathfrak{g}^{k+1} \subset \mathfrak{g}^k$, and the Lie algebra $\mathfrak{g}$ is said to be solvable if the derived series eventually arrives at the zero subalgebra, i.e. there exists the smallest natural number m such that $\mathfrak{g}^{m+1} = \{0\}$ or, in other words, $\mathfrak{g}^m$ is Abelian.
- Example. The Lie algebra of upper-triangular matrices is solvable.



Lie's Theorem

- A classical example is the celebrated result due to Lie, who established the following theorem :

Theorem

If n vector fields, $X_1,\ldots,X_n$, which are linearly independent at each point of an open set $U \subset \mathbb{R}^n$, span a solvable Lie algebra and satisfy $[X_1, X_i] = \lambda_i X_1$ with $\lambda_i \in \mathbb{R}$, then X_1 is integrable by quadratures in U.

- A different result is due to Kozlov.

Theorem

Let vector fields, $X_1,\ldots,X_n$, be linearly independent at each point of an open set $U \subset \mathbb{R}^n$ and span a Lie algebra L such that the corresponding operators of the adjoint representation $ad_{X_i} = [X_i, \cdot]$ have a common triangular form

$$[X_i, X_j] = \sum_{k=1}^{i} C_{ij}{}^k X_k, \qquad C_{ij}{}^k \in \mathbb{R}.$$

Then, all the vector fields X_i, $i = 1,\ldots,n$, are integrable by quadratures.

Lie's Theorem

- A classical example is the celebrated result due to Lie, who established the following theorem :

Theorem

If n vector fields, $X_1, \ldots, X_n$, which are linearly independent at each point of an open set $U \subset \mathbb{R}^n$, span a solvable Lie algebra and satisfy $[X_1, X_i] = \lambda_i X_1$ with $\lambda_i \in \mathbb{R}$, then X_1 is integrable by quadratures in U.

- A different result is due to Kozlov.

Theorem

Let vector fields, $X_1, \ldots, X_n$, be linearly independent at each point of an open set $U \subset \mathbb{R}^n$ and span a Lie algebra L such that the corresponding operators of the adjoint representation $ad_{X_i} = [X_i, \cdot]$ have a common triangular form

$$[X_i, X_j] = \sum_{k=1}^{i} C_{ij}{}^k X_k, \qquad C_{ij}{}^k \in \mathbb{R}.$$

Then, all the vector fields X_i, $i = 1, \ldots, n$, are integrable by quadratures.

Lie's Theorem

- A classical example is the celebrated result due to Lie, who established the following theorem :

Theorem

If n vector fields, $X_1,\ldots,X_n$, which are linearly independent at each point of an open set $U \subset \mathbb{R}^n$, span a solvable Lie algebra and satisfy $[X_1, X_i] = \lambda_i X_1$ with $\lambda_i \in \mathbb{R}$, then X_1 is integrable by quadratures in U.

- A different result is due to Kozlov.

Theorem

Let vector fields, $X_1,\ldots,X_n$, be linearly independent at each point of an open set $U \subset \mathbb{R}^n$ and span a Lie algebra L such that the corresponding operators of the adjoint representation $ad_{X_i} = [X_i, \cdot]$ have a common triangular form

$$[X_i, X_j] = \sum_{k=1}^{i} C_{ij}{}^k X_k, \qquad C_{ij}{}^k \in \mathbb{R}.$$

Then, all the vector fields X_i, $i = 1, \ldots, n$, are integrable by quadratures.

Lie's Theorem

- A classical example is the celebrated result due to Lie, who established the following theorem :

Theorem

If n vector fields, $X_1,\ldots,X_n$, which are linearly independent at each point of an open set $U \subset \mathbb{R}^n$, span a solvable Lie algebra and satisfy $[X_1, X_i] = \lambda_i X_1$ with $\lambda_i \in \mathbb{R}$, then X_1 is integrable by quadratures in U.

- A different result is due to Kozlov.

Theorem

Let vector fields, $X_1,\ldots,X_n$, be linearly independent at each point of an open set $U \subset \mathbb{R}^n$ and span a Lie algebra L such that the corresponding operators of the adjoint representation $ad_{X_i} = [X_i, \cdot]$ have a common triangular form

$$[X_i, X_j] = \sum_{k=1}^{i} C_{ij}{}^k X_k, \qquad C_{ij}{}^k \in \mathbb{R}.$$

Then, all the vector fields X_i, $i = 1,\ldots,n$, are integrable by quadratures.

Lie's Theorem

- A classical example is the celebrated result due to Lie, who established the following theorem :

Theorem

If n vector fields, $X_1,\ldots,X_n$, which are linearly independent at each point of an open set $U \subset \mathbb{R}^n$, span a solvable Lie algebra and satisfy $[X_1, X_i] = \lambda_i X_1$ with $\lambda_i \in \mathbb{R}$, then X_1 is integrable by quadratures in U.

- A different result is due to Kozlov.

Theorem

Let vector fields, $X_1,\ldots,X_n$, be linearly independent at each point of an open set $U \subset \mathbb{R}^n$ and span a Lie algebra L such that the corresponding operators of the adjoint representation $ad_{X_i} = [X_i, \cdot]$ have a common triangular form

$$[X_i, X_j] = \sum_{k=1}^{i} C_{ij}{}^k X_k, \qquad C_{ij}{}^k \in \mathbb{R}.$$

Then, all the vector fields X_i, $i = 1, \ldots, n$, are integrable by quadratures.

Sketch of the Lie's proof for $n = 2$

- The differential equation can be integrated if we are able to find a first integral F for X_1, i.e. $X_1 F = 0$, such that $dF \neq 0$ in U.
- As X_1 and X_2 are two linearly independent vector fields such that $[X_1, X_2] = \lambda_2 X_1$, there exists a 1-form α_0 such that $i(X_1)\alpha_0 = 0$, $i(X_2)\alpha = 1$.
- We can see that α is then closed, because X_1 and X_2 generate $\mathfrak{X}(\mathbb{R}^2)$ and

$$d\alpha(X_1, X_2) = X_1\alpha(X_2) - X_2\alpha(X_1) + \alpha([X_1, X_2]) = \alpha([X_1, X_2]) = \lambda_2\, \alpha(X_1) = 0$$

- The (locally defined) function F such that

$$F(x^1, x^2) = \int_{\gamma_{(x^1, x^2)}} \alpha,$$

where $\gamma_{(x^1,x^2)}$ is any curve joining a reference point $(x_0^1, x_0^2) \in U$ with the point (x^1, x^2), is the first integral we were looking for.

Sketch of the Lie's proof for $n = 2$

- The differential equation can be integrated if we are able to find a first integral F for X_1, i.e. $X_1 F = 0$, such that $dF \neq 0$ in U.
- As X_1 and X_2 are two linearly independent vector fields such that $[X_1, X_2] = \lambda_2 X_1$, there exists a 1-form α_0 such that $i(X_1)\alpha_0 = 0$, $i(X_2)\alpha = 1$.
- We can see that α is then closed, because X_1 and X_2 generate $\mathfrak{X}(\mathbb{R}^2)$ and

$$d\alpha(X_1, X_2) = X_1\alpha(X_2) - X_2\alpha(X_1) + \alpha([X_1, X_2]) = \alpha([X_1, X_2]) = \lambda_2\, \alpha(X_1) = 0$$

- The (locally defined) function F such that

$$F(x^1, x^2) = \int_{\gamma_{(x^1, x^2)}} \alpha,$$

where $\gamma_{(x^1, x^2)}$ is any curve joining a reference point $(x_0^1, x_0^2) \in U$ with the point (x^1, x^2), is the first integral we were looking for.

Sketch of the Lie's proof for $n = 2$

- The differential equation can be integrated if we are able to find a first integral F for X_1, i.e. $X_1 F = 0$, such that $dF \neq 0$ in U.
- As X_1 and X_2 are two linearly independent vector fields such that $[X_1, X_2] = \lambda_2 X_1$, there exists a 1-form α_0 such that $i(X_1)\alpha_0 = 0$, $i(X_2)\alpha = 1$.
- We can see that α is then closed, because X_1 and X_2 generate $\mathfrak{X}(\mathbb{R}^2)$ and

$$d\alpha(X_1, X_2) = X_1\alpha(X_2) - X_2\alpha(X_1) + \alpha([X_1, X_2]) = \alpha([X_1, X_2]) = \lambda_2\, \alpha(X_1) = 0$$

- The (locally defined) function F such that

$$F(x^1, x^2) = \int_{\gamma_{(x^1,x^2)}} \alpha,$$

where $\gamma_{(x^1,x^2)}$ is any curve joining a reference point $(x_0^1, x_0^2) \in U$ with the point (x^1, x^2), is the first integral we were looking for.

Sketch of the Lie's proof for $n = 2$

- The differential equation can be integrated if we are able to find a first integral F for X_1, i.e. $X_1 F = 0$, such that $dF \neq 0$ in U.
- As X_1 and X_2 are two linearly independent vector fields such that $[X_1, X_2] = \lambda_2 X_1$, there exists a 1-form α_0 such that $i(X_1)\alpha_0 = 0$, $i(X_2)\alpha = 1$.
- We can see that α is then closed, because X_1 and X_2 generate $\mathfrak{X}(\mathbb{R}^2)$ and

$$d\alpha(X_1, X_2) = X_1\alpha(X_2) - X_2\alpha(X_1) + \alpha([X_1, X_2]) = \alpha([X_1, X_2]) = \lambda_2\, \alpha(X_1) = 0$$

- The (locally defined) function F such that

$$F(x^1, x^2) = \int_{\gamma_{(x^1, x^2)}} \alpha,$$

where $\gamma_{(x^1,x^2)}$ is any curve joining a reference point $(x_0^1, x_0^2) \in U$ with the point (x^1, x^2), is the first integral we were looking for.

Sketch of the Lie's proof for $n = 2$

- The differential equation can be integrated if we are able to find a first integral F for X_1, i.e. $X_1 F = 0$, such that $dF \neq 0$ in U.
- As X_1 and X_2 are two linearly independent vector fields such that $[X_1, X_2] = \lambda_2\, X_1$, there exists a 1-form α_0 such that $i(X_1)\alpha_0 = 0$, $i(X_2)\alpha = 1$.
- We can see that α is then closed, because X_1 and X_2 generate $\mathfrak{X}(\mathbb{R}^2)$ and

$$d\alpha(X_1, X_2) = X_1\alpha(X_2) - X_2\alpha(X_1) + \alpha([X_1, X_2]) = \alpha([X_1, X_2]) = \lambda_2\, \alpha(X_1) = 0$$

- The (locally defined) function F such that

$$F(x^1, x^2) = \int_{\gamma_{(x^1,x^2)}} \alpha,$$

where $\gamma_{(x^1,x^2)}$ is any curve joining a reference point $(x_0^1, x_0^2) \in U$ with the point (x^1, x^2), is the first integral we were looking for.

Example

- The dynamics is given by the vector field X_1, defined in $M = \mathrm{T}^*\mathbb{R}^2 = \mathbb{R}^2 \times \mathbb{R}^2$ with coordinates (x, y, p_x, p_y), by

$$X_1 = p_x\partial_x + p_y\partial_y - \frac{k_2}{y^{2/3}}\partial_{p_x} + \frac{2}{3}\frac{k_2\,x + k_3}{y^{5/3}}\partial_{p_y}\,,$$

  where k_2 and k_3 are arbitrary constants.
- Now, with X_i, $i = 2, 3, 4$, we denote the vector fields

$$\begin{aligned} X_2 &= \left(6\,p_x^2 + 3\,p_y^2 + k_2\frac{6x}{y^{2/3}} + k_3\frac{6}{y^{2/3}}\right)\partial_x + (6\,p_x p_y + 9\,k_2 y^{1/3})\partial_y \\ &- k_2\frac{6}{y^{2/3}}\,p_x\partial_{p_x} + \left(4k_2\frac{x}{y^{5/3}} - 3\frac{1}{y^{2/3}}\,p_y\right)\partial_{p_y}\,, \end{aligned}$$

Example

- The dynamics is given by the vector field X_1, defined in $M = \mathrm{T}^*\mathbb{R}^2 = \mathbb{R}^2 \times \mathbb{R}^2$ with coordinates (x, y, p_x, p_y), by

$$X_1 = p_x\partial_x + p_y\partial_y - \frac{k_2}{y^{2/3}}\partial_{p_x} + \frac{2}{3}\frac{k_2\, x + k_3}{y^{5/3}}\partial_{p_y}\,,$$

where k_2 and k_3 are arbitrary constants.

- Now, with X_i, $i = 2, 3, 4$, we denote the vector fields

$$\begin{aligned} X_2 &= \left(6\,p_x^2 + 3\,p_y^2 + k_2\frac{6x}{y^{2/3}} + k_3\frac{6}{y^{2/3}}\right)\partial_x + (6\,p_x p_y + 9\,k_2 y^{1/3})\partial_y \\ &- k_2\frac{6}{y^{2/3}}\,p_x\partial_{p_x} + \left(4k_2\frac{x}{y^{5/3}} - 3\frac{1}{y^{2/3}}\,p_y\right)\partial_{p_y}\,, \end{aligned}$$

Example

- The dynamics is given by the vector field X_1, defined in $M = \mathrm{T}^*\mathbb{R}^2 = \mathbb{R}^2 \times \mathbb{R}^2$ with coordinates (x, y, p_x, p_y), by

$$X_1 = p_x\partial_x + p_y\partial_y - \frac{k_2}{y^{2/3}}\partial_{p_x} + \frac{2}{3}\frac{k_2\,x + k_3}{y^{5/3}}\partial_{p_y}\,,$$

  where k_2 and k_3 are arbitrary constants.

- Now, with X_i, $i = 2, 3, 4$, we denote the vector fields

$$\begin{aligned} X_2 \;\; &= \;\; \left(6\,p_x^2 + 3\,p_y^2 + k_2\frac{6x}{y^{2/3}} + k_3\frac{6}{y^{2/3}}\right)\partial_x + (6\,p_xp_y + 9\,k_2y^{1/3})\partial_y \\ &- \;\; k_2\frac{6}{y^{2/3}}\,p_x\partial_{p_x} + \left(4k_2\frac{x}{y^{5/3}} - 3\frac{1}{y^{2/3}}\,p_y\right)\partial_{p_y}\,, \end{aligned}$$

Example

$$
\begin{aligned}
X_3 \;&=\; \left(4\,p_x^3+4\,p_xp_y^2+\frac{8(k_2x+k_3)}{y^{2/3}}p_x+12k_2\,y^{1/3}\,p_y\right)\partial_x\\
&+\;\left(4p_x^2\,p_y+12k_2\,y^{1/3}\,p_x\right)\partial_y-4k_2\frac{1}{y^{2/3}}p_x^2\,\partial_{p_x}\\
&+\;\left(\frac{8}{3}\frac{k_2x+k_3}{y^{5/3}}p_x^2-4k_2\frac{1}{y^{2/3}}p_xp_y-12\,k_2^2\frac{1}{y^{1/3}}\right)\partial_{p_y}\,,
\end{aligned}
$$

and

$$
\begin{gathered}
X_4=\\
\left(6p_x^5+12\,p_x^3p_y^2+24\frac{k_3+k_2x}{y^{2/3}}p_x^3+108\,k_2y^{1/3}p_x^2p_y+324\,k_2^2y^{2/3}p_x\right)\partial_x\\
+\left(6\,p_x^4p_y+36\,k_2y^{1/3}p_x^3\right)\partial_y-6\,\left(\frac{k_2}{y^{2/3}}p_x^4-972k_2^3\right)\partial_{p_x}\\
+\left(4\,\frac{k_3+k_2x}{y^{5/3}}p_x^4-12\frac{k_2}{y^{2/3}}-108\,k_2^2\frac{1}{y^{1/3}}p_x^2\right)\partial_{p_y}\,.
\end{gathered}
$$

Example

$$
\begin{aligned}
X_3 \;=\; & \left(4\, p_x^3 + 4\, p_x p_y^2 + \frac{8(k_2 x + k_3)}{y^{2/3}} p_x + 12 k_2\, y^{1/3}\, p_y\right) \partial_x \\
+ \; & \left(4 p_x^2\, p_y + 12 k_2\, y^{1/3}\, p_x\right) \partial_y - 4 k_2 \frac{1}{y^{2/3}} p_x^2\, \partial_{p_x} \\
+ \; & \left(\frac{8}{3} \frac{k_2 x + k_3}{y^{5/3}} p_x^2 - 4 k_2 \frac{1}{y^{2/3}} p_x p_y - 12\, k_2^2 \frac{1}{y^{1/3}}\right) \partial_{p_y} \,,
\end{aligned}
$$

and

$$
\begin{aligned}
X_4 = & \\
& \left(6 p_x^5 + 12\, p_x^3 p_y^2 + 24 \frac{k_3 + k_2 x}{y^{2/3}} p_x^3 + 108\, k_2 y^{1/3} p_x^2 p_y + 324\, k_2^2 y^{2/3} p_x\right) \partial_x \\
& + \left(6\, p_x^4 p_y + 36\, k_2 y^{1/3} p_x^3\right) \partial_y - 6\, \left(\frac{k_2}{y^{2/3}} p_x^4 - 972 k_2^3\right) \partial_{p_x} \\
& + \left(4\, \frac{k_3 + k_2 x}{y^{5/3}} p_x^4 - 12 \frac{k_2}{y^{2/3}} - 108\, k_2^2 \frac{1}{y^{1/3}} p_x^2\right) \partial_{p_y} \,.
\end{aligned}
$$

Example

$$
\begin{aligned}
X_3 &= \left(4\,p_x^3 + 4\,p_x p_y^2 + \frac{8(k_2 x + k_3)}{y^{2/3}} p_x + 12 k_2\, y^{1/3}\, p_y\right) \partial_x \\
&+ \left(4 p_x^2\, p_y + 12 k_2\, y^{1/3}\, p_x\right) \partial_y - 4 k_2 \frac{1}{y^{2/3}} p_x^2\, \partial_{p_x} \\
&+ \left(\frac{8}{3} \frac{k_2 x + k_3}{y^{5/3}} p_x^2 - 4 k_2 \frac{1}{y^{2/3}} p_x p_y - 12\, k_2^2 \frac{1}{y^{1/3}}\right) \partial_{p_y}\,,
\end{aligned}
$$

and

$$
\begin{gathered}
X_4 = \\
\left(6 p_x^5 + 12\, p_x^3 p_y^2 + 24 \frac{k_3 + k_2 x}{y^{2/3}} p_x^3 + 108\, k_2 y^{1/3} p_x^2 p_y + 324\, k_2^2 y^{2/3} p_x\right) \partial_x \\
+ \left(6\, p_x^4 p_y + 36\, k_2 y^{1/3} p_x^3\right) \partial_y - 6\, \left(\frac{k_2}{y^{2/3}} p_x^4 - 972 k_2^3\right) \partial_{p_x} \\
+ \left(4\, \frac{k_3 + k_2 x}{y^{5/3}} p_x^4 - 12 \frac{k_2}{y^{2/3}} - 108\, k_2^2 \frac{1}{y^{1/3}} p_x^2\right) \partial_{p_y}\,.
\end{gathered}
$$

Example

- Then, we have

$$[X_1, X_i] = 0\,, \quad i = 2, 3, 4.$$

- $$[X_2, X_3] = 0\,, \qquad [X_2, X_4] = 1944\,k_2^3\,X_1\,, \qquad [X_3, X_4] = 432\,k_2^3\,X_2\,.$$

- Therefore, X_1, X_2, X_3, X_4 generate a four-dimensional solvable real Lie algebra L and are linearly independent in $\mathbb{R}^4$.
- In view of Lie's Theorem,

$$X_1 = p_x\partial_x + p_y\partial_y - \frac{k_2}{y^{2/3}}\partial_{p_x} + \frac{2}{3}\frac{k_2\,x + k_3}{y^{5/3}}\partial_{p_y}\,,$$

is integrable by quadratures

- and in view of Kozlov's Theorem all X_1, X_2, X_3, X_4 are integrable by quadratures.

Example

- Then, we have
$$[X_1, X_i] = 0\,, \quad i = 2,3,4.$$
-
$$[X_2, X_3] = 0\,, \qquad [X_2, X_4] = 1944\, k_2^3\, X_1\,, \qquad [X_3, X_4] = 432\, k_2^3\, X_2\,.$$
- Therefore, X_1, X_2, X_3, X_4 generate a four-dimensional solvable real Lie algebra L and are linearly independent in $\mathbb{R}^4$.
- In view of Lie's Theorem,
$$X_1 = p_x\partial_x + p_y\partial_y - \frac{k_2}{y^{2/3}}\partial_{p_x} + \frac{2}{3}\frac{k_2\,x + k_3}{y^{5/3}}\partial_{p_y}\,,$$
is integrable by quadratures
- and in view of Kozlov's Theorem all X_1, X_2, X_3, X_4 are integrable by quadratures.

Example

- Then, we have
$$[X_1, X_i] = 0\,, \quad i = 2, 3, 4.$$
-
$$[X_2, X_3] = 0\,, \qquad [X_2, X_4] = 1944\, k_2^3\, X_1\,, \qquad [X_3, X_4] = 432\, k_2^3\, X_2\,.$$
- Therefore, X_1, X_2, X_3, X_4 generate a four-dimensional solvable real Lie algebra L and are linearly independent in $\mathbb{R}^4$.
- In view of Lie's Theorem,
$$X_1 = p_x\partial_x + p_y\partial_y - \frac{k_2}{y^{2/3}}\partial_{p_x} + \frac{2}{3}\frac{k_2\, x + k_3}{y^{5/3}}\partial_{p_y}\,,$$
is integrable by quadratures
- and in view of Kozlov's Theorem all X_1, X_2, X_3, X_4 are integrable by quadratures.

Example

- Then, we have

$$[X_1, X_i] = 0\,, \quad i = 2, 3, 4.$$

-

$$[X_2, X_3] = 0\,, \qquad [X_2, X_4] = 1944\, k_2^3\, X_1\,, \qquad [X_3, X_4] = 432\, k_2^3\, X_2\,.$$

- Therefore, X_1, X_2, X_3, X_4 generate a four-dimensional solvable real Lie algebra L and are linearly independent in $\mathbb{R}^4$.
- In view of Lie's Theorem,

$$X_1 = p_x\partial_x + p_y\partial_y - \frac{k_2}{y^{2/3}}\partial_{p_x} + \frac{2}{3}\frac{k_2\,x + k_3}{y^{5/3}}\partial_{p_y}\,,$$

is integrable by quadratures

- and in view of Kozlov's Theorem all X_1, X_2, X_3, X_4 are integrable by quadratures.

Example

- Then, we have
$$[X_1, X_i] = 0\,, \quad i = 2, 3, 4.$$
-
$$[X_2, X_3] = 0\,, \qquad [X_2, X_4] = 1944\, k_2^3\, X_1\,, \qquad [X_3, X_4] = 432\, k_2^3\, X_2\,.$$
- Therefore, X_1, X_2, X_3, X_4 generate a four-dimensional solvable real Lie algebra L and are linearly independent in $\mathbb{R}^4$.
- In view of Lie's Theorem,
$$X_1 = p_x\partial_x + p_y\partial_y - \frac{k_2}{y^{2/3}}\partial_{p_x} + \frac{2}{3}\frac{k_2\, x + k_3}{y^{5/3}}\partial_{p_y}\,,$$
is integrable by quadratures
- and in view of Kozlov's Theorem all X_1, X_2, X_3, X_4 are integrable by quadratures.

Example

- Then, we have
$$[X_1, X_i] = 0\,, \quad i = 2, 3, 4.$$
- $$[X_2, X_3] = 0\,, \qquad [X_2, X_4] = 1944\, k_2^3\, X_1\,, \qquad [X_3, X_4] = 432\, k_2^3\, X_2\,.$$
- Therefore, X_1, X_2, X_3, X_4 generate a four-dimensional solvable real Lie algebra L and are linearly independent in $\mathbb{R}^4$.
- In view of Lie's Theorem,
$$X_1 = p_x\partial_x + p_y\partial_y - \frac{k_2}{y^{2/3}}\partial_{p_x} + \frac{2}{3}\frac{k_2\, x + k_3}{y^{5/3}}\partial_{p_y}\,,$$
is integrable by quadratures
- and in view of Kozlov's Theorem all X_1, X_2, X_3, X_4 are integrable by quadratures.

Main result

- We want to generalize the mentioned theorems of Lie and Kozlow on a finite-dimansional solvable Lie algebra L of vector fields on M by
- skipping the assumption that the dimension of L equals $\dim(M)$,
- skipping the triangularizability assumption.
- Hence, our main result can be formulated as follows.

Theorem

If L is a finite-dimensional solvable and transitive real Lie algebra of vector fields on a manifold M, then each vector field $\Gamma \in L$ is integrable by quadratures.

- We will proceed by induction on $n = \dim(M)$ using the following lemma.

Lemma

Any solvable finite-dimensional real Lie algebra L contains an Abelian Lie ideal A of dimension 1 or 2.

Main result

- We want to generalize the mentioned theorems of Lie and Kozlow on a finite-dimansional solvable Lie algebra L of vector fields on M by
- skipping the assumption that the dimension of L equals $\dim(M)$,
- skipping the triangularizability assumption.
- Hence, our main result can be formulated as follows.

Theorem

If L is a finite-dimensional solvable and transitive real Lie algebra of vector fields on a manifold M, then each vector field $\Gamma \in L$ is integrable by quadratures.

- We will proceed by induction on $n = \dim(M)$ using the following lemma.

Lemma

Any solvable finite-dimensional real Lie algebra L contains an Abelian Lie ideal A of dimension 1 or 2.

Main result

- We want to generalize the mentioned theorems of Lie and Kozlow on a finite-dimansional solvable Lie algebra L of vector fields on M by
- skipping the assumption that the dimension of L equals $\dim(M)$,
- skipping the triangularizability assumption.
- Hence, our main result can be formulated as follows.

Theorem

If L is a finite-dimensional solvable and transitive real Lie algebra of vector fields on a manifold M, then each vector field $\Gamma \in L$ is integrable by quadratures.

- We will proceed by induction on $n = \dim(M)$ using the following lemma.

Lemma

Any solvable finite-dimensional real Lie algebra L contains an Abelian Lie ideal A of dimension 1 or 2.

Main result

- We want to generalize the mentioned theorems of Lie and Kozlow on a finite-dimansional solvable Lie algebra L of vector fields on M by
- skipping the assumption that the dimension of L equals $\dim(M)$,
- skipping the triangularizability assumption.
- Hence, our main result can be formulated as follows.

Theorem

If L is a finite-dimensional solvable and transitive real Lie algebra of vector fields on a manifold M, then each vector field $\Gamma \in L$ is integrable by quadratures.

- We will proceed by induction on $n = \dim(M)$ using the following lemma.

Lemma

Any solvable finite-dimensional real Lie algebra L contains an Abelian Lie ideal A of dimension 1 or 2.

Main result

- We want to generalize the mentioned theorems of Lie and Kozlow on a finite-dimansional solvable Lie algebra L of vector fields on M by
- skipping the assumption that the dimension of L equals $\dim(M)$,
- skipping the triangularizability assumption.
- Hence, our main result can be formulated as follows.

Theorem

If L is a finite-dimensional solvable and transitive real Lie algebra of vector fields on a manifold M, then each vector field $\Gamma \in L$ is integrable by quadratures.

- We will proceed by induction on $n = \dim(M)$ using the following lemma.

Lemma

Any solvable finite-dimensional real Lie algebra L contains an Abelian Lie ideal A of dimension 1 or 2.

Main result

- We want to generalize the mentioned theorems of Lie and Kozlow on a finite-dimansional solvable Lie algebra L of vector fields on M by
- skipping the assumption that the dimension of L equals $\dim(M)$,
- skipping the triangularizability assumption.
- Hence, our main result can be formulated as follows.

Theorem

If L is a finite-dimensional solvable and transitive real Lie algebra of vector fields on a manifold M, then each vector field $\Gamma \in L$ is integrable by quadratures.

- We will proceed by induction on $n = \dim(M)$ using the following lemma.

Lemma

Any solvable finite-dimensional real Lie algebra L contains an Abelian Lie ideal A of dimension 1 or 2.

Main result

- We want to generalize the mentioned theorems of Lie and Kozlow on a finite-dimansional solvable Lie algebra L of vector fields on M by
- skipping the assumption that the dimension of L equals $\dim(M)$,
- skipping the triangularizability assumption.
- Hence, our main result can be formulated as follows.

Theorem

If L is a finite-dimensional solvable and transitive real Lie algebra of vector fields on a manifold M, then each vector field $\Gamma \in L$ is integrable by quadratures.

- We will proceed by induction on $n = \dim(M)$ using the following lemma.

Lemma

Any solvable finite-dimensional real Lie algebra L contains an Abelian Lie ideal A of dimension 1 or 2.

Main result

- We want to generalize the mentioned theorems of Lie and Kozlow on a finite-dimansional solvable Lie algebra L of vector fields on M by
- skipping the assumption that the dimension of L equals $\dim(M)$,
- skipping the triangularizability assumption.
- Hence, our main result can be formulated as follows.

Theorem

If L is a finite-dimensional solvable and transitive real Lie algebra of vector fields on a manifold M, then each vector field $\Gamma \in L$ is integrable by quadratures.

- We will proceed by induction on $n = \dim(M)$ using the following lemma.

Lemma

Any solvable finite-dimensional real Lie algebra L contains an Abelian Lie ideal A of dimension 1 or 2.

Abelian Lie ideals

Proof.

- Another important Lie theorem ensures that every finite-dimensional representation of a solvable Lie algebra over an algebraically closed field has an eigenvector common to all the operators of the representation.
- If we consider the adjoint representation, the theorem implies that any finite-dimensional complex, solvable Lie algebra has a one dimensional ideal.
- Therefore, we can consider the complexified Lie algebra $L^{\mathbb{C}} = L \oplus iL$ and its adjoint representation for which we can use the standard Lie theorem. As there is a common eigenvector $\nu = \nu_1 + i\nu_2$, the vectors $\nu_1, \nu_2 \in L$ span an Abelian Lie ideal A of dimension 1 or 2.
- Indeed, A is clearly a Lie ideal, $[x, \nu] = [x, \nu_1] + i[x, \nu_2] = \lambda(x)\nu$, and if $\dim(A) = 2$, then $[\nu_2, \nu_1] = [\nu_2, \nu] = \lambda\nu$, so $\lambda = 0$.

□

Abelian Lie ideals

Proof.

- Another important Lie theorem ensures that every finite-dimensional representation of a solvable Lie algebra over an algebraically closed field has an eigenvector common to all the operators of the representation.
- If we consider the adjoint representation, the theorem implies that any finite-dimensional complex, solvable Lie algebra has a one dimensional ideal.
- Therefore, we can consider the complexified Lie algebra $L^{\mathbb{C}} = L \oplus iL$ and its adjoint representation for which we can use the standard Lie theorem. As there is a common eigenvector $\nu = \nu_1 + i\nu_2$, the vectors $\nu_1, \nu_2 \in L$ span an Abelian Lie ideal A of dimension 1 or 2.
- Indeed, A is clearly a Lie ideal, $[x, \nu] = [x, \nu_1] + i[x, \nu_2] = \lambda(x)\nu$, and if $\dim(A) = 2$, then $[\nu_2, \nu_1] = [\nu_2, \nu] = \lambda\nu$, so $\lambda = 0$.

□

Abelian Lie ideals

Proof.

- Another important Lie theorem ensures that every finite-dimensional representation of a solvable Lie algebra over an algebraically closed field has an eigenvector common to all the operators of the representation.
- If we consider the adjoint representation, the theorem implies that any finite-dimensional complex, solvable Lie algebra has a one dimensional ideal.
- Therefore, we can consider the complexified Lie algebra $L^{\mathbb{C}} = L \oplus iL$ and its adjoint representation for which we can use the standard Lie theorem. As there is a common eigenvector $\nu = \nu_1 + i\nu_2$, the vectors $\nu_1, \nu_2 \in L$ span an Abelian Lie ideal A of dimension 1 or 2.
- Indeed, A is clearly a Lie ideal, $[x, \nu] = [x, \nu_1] + i[x, \nu_2] = \lambda(x)\nu$, and if $\dim(A) = 2$, then $[\nu_2, \nu_1] = [\nu_2, \nu] = \lambda\nu$, so $\lambda = 0$.

Abelian Lie ideals

Proof.

- Another important Lie theorem ensures that every finite-dimensional representation of a solvable Lie algebra over an algebraically closed field has an eigenvector common to all the operators of the representation.
- If we consider the adjoint representation, the theorem implies that any finite-dimensional complex, solvable Lie algebra has a one dimensional ideal.
- Therefore, we can consider the complexified Lie algebra $L^{\mathbb{C}} = L \oplus iL$ and its adjoint representation for which we can use the standard Lie theorem. As there is a common eigenvector $\nu = \nu_1 + i\nu_2$, the vectors $\nu_1, \nu_2 \in L$ span an Abelian Lie ideal A of dimension 1 or 2.
- Indeed, A is clearly a Lie ideal, $[x, \nu] = [x, \nu_1] + i[x, \nu_2] = \lambda(x)\nu$, and if $\dim(A) = 2$, then $[\nu_2, \nu_1] = [\nu_2, \nu] = \lambda\nu$, so $\lambda = 0$.

□

Abelian Lie ideals

Proof.

- Another important Lie theorem ensures that every finite-dimensional representation of a solvable Lie algebra over an algebraically closed field has an eigenvector common to all the operators of the representation.
- If we consider the adjoint representation, the theorem implies that any finite-dimensional complex, solvable Lie algebra has a one dimensional ideal.
- Therefore, we can consider the complexified Lie algebra $L^{\mathbb{C}} = L \oplus iL$ and its adjoint representation for which we can use the standard Lie theorem. As there is a common eigenvector $\nu = \nu_1 + i\nu_2$, the vectors $\nu_1, \nu_2 \in L$ span an Abelian Lie ideal A of dimension 1 or 2.
- Indeed, A is clearly a Lie ideal, $[x, \nu] = [x, \nu_1] + i[x, \nu_2] = \lambda(x)\nu$, and if $\dim(A) = 2$, then $[\nu_2, \nu_1] = [\nu_2, \nu] = \lambda\nu$, so $\lambda = 0$.

□

Abelian Lie ideals

Proof.

- Another important Lie theorem ensures that every finite-dimensional representation of a solvable Lie algebra over an algebraically closed field has an eigenvector common to all the operators of the representation.
- If we consider the adjoint representation, the theorem implies that any finite-dimensional complex, solvable Lie algebra has a one dimensional ideal.
- Therefore, we can consider the complexified Lie algebra $L^{\mathbb{C}} = L \oplus iL$ and its adjoint representation for which we can use the standard Lie theorem. As there is a common eigenvector $\nu = \nu_1 + i\nu_2$, the vectors $\nu_1, \nu_2 \in L$ span an Abelian Lie ideal A of dimension 1 or 2.
- Indeed, A is clearly a Lie ideal, $[x, \nu] = [x, \nu_1] + i[x, \nu_2] = \lambda(x)\nu$, and if $\dim(A) = 2$, then $[\nu_2, \nu_1] = [\nu_2, \nu] = \lambda\nu$, so $\lambda = 0$.

□

Abelian Lie ideals

Proof.

- Another important Lie theorem ensures that every finite-dimensional representation of a solvable Lie algebra over an algebraically closed field has an eigenvector common to all the operators of the representation.
- If we consider the adjoint representation, the theorem implies that any finite-dimensional complex, solvable Lie algebra has a one dimensional ideal.
- Therefore, we can consider the complexified Lie algebra $L^{\mathbb{C}} = L \oplus iL$ and its adjoint representation for which we can use the standard Lie theorem. As there is a common eigenvector $\nu = \nu_1 + i\nu_2$, the vectors $\nu_1, \nu_2 \in L$ span an Abelian Lie ideal A of dimension 1 or 2.
- Indeed, A is clearly a Lie ideal, $[x, \nu] = [x, \nu_1] + i[x, \nu_2] = \lambda(x)\nu$, and if $\dim(A) = 2$, then $[\nu_2, \nu_1] = [\nu_2, \nu] = \lambda\nu$, so $\lambda = 0$.

□

Rectification of Abelian algebras of vector fields

Definition

We say that an Abelian subalgebra of vector fields A is straightened out, (or rectified) by quadratures in an open set U if, by quadratures, we can find local coordinates $(Q^1, \ldots, Q^n)$ in U such that the set $\{\partial_{Q^1}, \ldots, \partial_{Q^r}\} \subset A$ and it generates the same distribution as A.

Proposition

Any Abelian ideal A of a transitive finite-dimensional solvable Lie algebra L of vector fields, can be straightened out by quadratures.

Proof.

Consider an Abelian ideal $A \lhd L$ and the descending series L_A^i of Lie ideals

$$L_A^0 = L, \quad L_A^{i+1} = [L_A^i, L_A^i] + A = L^{i+1} + A\,,\ i \in \mathbb{N}\,. \tag{2}$$

□

Rectification of Abelian algebras of vector fields

Definition

We say that an Abelian subalgebra of vector fields A is straightened out, (or rectified) by quadratures in an open set U if, by quadratures, we can find local coordinates $(Q^1, \ldots, Q^n)$ in U such that the set $\{\partial_{Q^1}, \ldots, \partial_{Q^r}\} \subset A$ and it generates the same distribution as A.

Proposition

Any Abelian ideal A of a transitive finite-dimensional solvable Lie algebra L of vector fields, can be straightened out by quadratures.

Proof.

Consider an Abelian ideal $A \lhd L$ and the descending series L_A^i of Lie ideals

$$L_A^0 = L, \quad L_A^{i+1} = [L_A^i, L_A^i] + A = L^{i+1} + A\,, \; i \in \mathbb{N}\,. \tag{2}$$

□

Rectification of Abelian algebras of vector fields

Definition

We say that an Abelian subalgebra of vector fields A is straightened out, (or rectified) by quadratures in an open set U if, by quadratures, we can find local coordinates $(Q^1, \ldots, Q^n)$ in U such that the set $\{\partial_{Q^1}, \ldots, \partial_{Q^r}\} \subset A$ and it generates the same distribution as A.

Proposition

Any Abelian ideal A of a transitive finite-dimensional solvable Lie algebra L of vector fields, can be straightened out by quadratures.

Proof.

Consider an Abelian ideal $A \lhd L$ and the descending series L_A^i of Lie ideals

$$L_A^0 = L, \quad L_A^{i+1} = [L_A^i, L_A^i] + A = L^{i+1} + A\,, \; i \in \mathbb{N}\,. \tag{2}$$

□

Rectification of Abelian algebras of vector fields

Definition

We say that an Abelian subalgebra of vector fields A is straightened out, (or rectified) by quadratures in an open set U if, by quadratures, we can find local coordinates $(Q^1, \dots, Q^n)$ in U such that the set $\{\partial_{Q^1}, \dots, \partial_{Q^r}\} \subset A$ and it generates the same distribution as A.

Proposition

Any Abelian ideal A of a transitive finite-dimensional solvable Lie algebra L of vector fields, can be straightened out by quadratures.

Proof.

Consider an Abelian ideal $A \lhd L$ and the descending series L_A^i of Lie ideals

$$L_A^0 = L, \quad L_A^{i+1} = [L_A^i, L_A^i] + A = L^{i+1} + A\,,\ i \in \mathbb{N}\,. \tag{2}$$

□

Rectification of Abelian algebras of vector fields

- As L is solvable, there is m such that $L_A^m = A$.
- Next we consider the generalised distribution $\mathcal{D}_A^i$ spanned by L_A^i. As L is transitive, it is actually a regular distribution. It is obviously (Frobenius) integrable, because L_A^i is a Lie algebra, so it is defining a foliation $\mathcal{F}_A^i$.
- Let r_i be the dimension of this foliation, $r_{i+1} \leq r_i$.
- There is a neighbourhood U of p and a coordinate system $(Q) = (Q^1, \dots, Q^n)$ therein, which can be obtained by quadratures from any given one, such that (Q) locally determines the series of foliations $\mathcal{F}_A^k$, $k = 0, 1, 2, \dots$, i.e.

$$\text{the leaves of} \quad \mathcal{F}_A^k \cap U \quad \text{are the level sets of} \quad Q^{r_k+1}, \dots, Q^n\,. \tag{3}$$

- Moreover, if $L_A^m = A$, then these coordinates can be chosen so that $\partial_{Q^1}, \dots, \partial_{Q^{r_m}}$ belong to A.

Rectification of Abelian algebras of vector fields

- As L is solvable, there is m such that $L_A^m = A$.
- Next we consider the generalised distribution $\mathcal{D}_A^i$ spanned by L_A^i. As L is transitive, it is actually a regular distribution. It is obviously (Frobenius) integrable, because L_A^i is a Lie algebra, so it is defining a foliation $\mathcal{F}_A^i$.
- Let r_i be the dimension of this foliation, $r_{i+1} \leq r_i$.
- There is a neighbourhood U of p and a coordinate system $(Q) = (Q^1, \ldots, Q^n)$ therein, which can be obtained by quadratures from any given one, such that (Q) locally determines the series of foliations $\mathcal{F}_A^k$, $k = 0, 1, 2, \ldots$, i.e.

$$\text{the leaves of} \quad \mathcal{F}_A^k \cap U \quad \text{are the level sets of} \quad Q^{r_k+1}, \ldots, Q^n\,. \tag{3}$$

- Moreover, if $L_A^m = A$, then these coordinates can be chosen so that $\partial_{Q^1}, \ldots, \partial_{Q^{r_m}}$ belong to A.

Rectification of Abelian algebras of vector fields

- As L is solvable, there is m such that $L_A^m = A$.
- Next we consider the generalised distribution $\mathcal{D}_A^i$ spanned by L_A^i. As L is transitive, it is actually a regular distribution. It is obviously (Frobenius) integrable, because L_A^i is a Lie algebra, so it is defining a foliation $\mathcal{F}_A^i$.
- Let r_i be the dimension of this foliation, $r_{i+1} \leq r_i$.
- There is a neighbourhood U of p and a coordinate system $(Q) = (Q^1, \ldots, Q^n)$ therein, which can be obtained by quadratures from any given one, such that (Q) locally determines the series of foliations $\mathcal{F}_A^k$, $k = 0, 1, 2, \ldots$, i.e.

$$\text{the leaves of} \quad \mathcal{F}_A^k \cap U \quad \text{are the level sets of} \quad Q^{r_k+1}, \ldots, Q^n\,. \tag{3}$$

- Moreover, if $L_A^m = A$, then these coordinates can be chosen so that $\partial_{Q^1}, \ldots, \partial_{Q^{r_m}}$ belong to A.

Rectification of Abelian algebras of vector fields

- As L is solvable, there is m such that $L_A^m = A$.
- Next we consider the generalised distribution $\mathcal{D}_A^i$ spanned by L_A^i. As L is transitive, it is actually a regular distribution. It is obviously (Frobenius) integrable, because L_A^i is a Lie algebra, so it is defining a foliation $\mathcal{F}_A^i$.
- Let r_i be the dimension of this foliation, $r_{i+1} \leq r_i$.
- There is a neighbourhood U of p and a coordinate system $(Q) = (Q^1, \dots, Q^n)$ therein, which can be obtained by quadratures from any given one, such that (Q) locally determines the series of foliations $\mathcal{F}_A^k$, $k = 0, 1, 2, \dots$, i.e.

$$\text{the leaves of} \quad \mathcal{F}_A^k \cap U \quad \text{are the level sets of} \quad Q^{r_k+1}, \dots, Q^n . \tag{3}$$

- Moreover, if $L_A^m = A$, then these coordinates can be chosen so that $\partial_{Q^1}, \dots, \partial_{Q^{r_m}}$ belong to A.

Rectification of Abelian algebras of vector fields

- As L is solvable, there is m such that $L_A^m = A$.
- Next we consider the generalised distribution $\mathcal{D}_A^i$ spanned by L_A^i. As L is transitive, it is actually a regular distribution. It is obviously (Frobenius) integrable, because L_A^i is a Lie algebra, so it is defining a foliation $\mathcal{F}_A^i$.
- Let r_i be the dimension of this foliation, $r_{i+1} \leq r_i$.
- There is a neighbourhood U of p and a coordinate system $(Q) = (Q^1, \ldots, Q^n)$ therein, which can be obtained by quadratures from any given one, such that (Q) locally determines the series of foliations $\mathcal{F}_A^k$, $k = 0, 1, 2, \ldots$, i.e.

$$\text{the leaves of} \quad \mathcal{F}_A^k \cap U \quad \text{are the level sets of} \quad Q^{r_k+1}, \ldots, Q^n. \quad (3)$$

- Moreover, if $L_A^m = A$, then these coordinates can be chosen so that $\partial_{Q^1}, \ldots, \partial_{Q^{r_m}}$ belong to A.

Rectification of Abelian algebras of vector fields

- As L is solvable, there is m such that $L_A^m = A$.
- Next we consider the generalised distribution $\mathcal{D}_A^i$ spanned by L_A^i. As L is transitive, it is actually a regular distribution. It is obviously (Frobenius) integrable, because L_A^i is a Lie algebra, so it is defining a foliation $\mathcal{F}_A^i$.
- Let r_i be the dimension of this foliation, $r_{i+1} \leq r_i$.
- There is a neighbourhood U of p and a coordinate system $(Q) = (Q^1, \ldots, Q^n)$ therein, which can be obtained by quadratures from any given one, such that (Q) locally determines the series of foliations $\mathcal{F}_A^k$, $k = 0, 1, 2, \ldots$, i.e.

$$\text{the leaves of} \quad \mathcal{F}_A^k \cap U \quad \text{are the level sets of} \quad Q^{r_k+1}, \ldots, Q^n . \qquad (3)$$

- Moreover, if $L_A^m = A$, then these coordinates can be chosen so that $\partial_{Q^1}, \ldots, \partial_{Q^{r_m}}$ belong to A.

Rectification of Abelian algebras of vector fields

- As L is solvable, there is m such that $L_A^m = A$.
- Next we consider the generalised distribution $\mathcal{D}_A^i$ spanned by L_A^i. As L is transitive, it is actually a regular distribution. It is obviously (Frobenius) integrable, because L_A^i is a Lie algebra, so it is defining a foliation $\mathcal{F}_A^i$.
- Let r_i be the dimension of this foliation, $r_{i+1} \leq r_i$.
- There is a neighbourhood U of p and a coordinate system $(Q) = (Q^1, \ldots, Q^n)$ therein, which can be obtained by quadratures from any given one, such that (Q) locally determines the series of foliations $\mathcal{F}_A^k$, $k = 0, 1, 2, \ldots$, i.e.

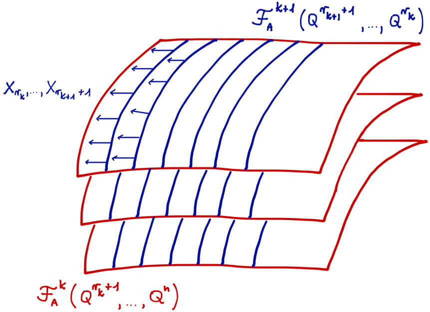
$$\text{the leaves of} \quad \mathcal{F}_A^k \cap U \quad \text{are the level sets of} \quad Q^{r_k+1}, \ldots, Q^n \,. \tag{3}$$

- Moreover, if $L_A^m = A$, then these coordinates can be chosen so that $\partial_{Q^1}, \ldots, \partial_{Q^{r_m}}$ belong to A.

Rectification of Abelian algebras of vector fields

- Inductively, suppose $Q^{r_k+1},\dots,Q^n$ are chosen in this way and let $X_{r_{k+1}+1},\dots,X_{r_k}\in L^k_A$ be chosen so that they span a vector subspace of $\mathcal{D}^k_A(p)$ complementary to $\mathcal{D}^{k+1}_A(p)$, i.e.

$$\langle X_{r_{k+1}+1}(p),\dots,X_{r_k}(p)\rangle\oplus\mathcal{D}^{k+1}_A(p)=\mathcal{D}^k(p).$$

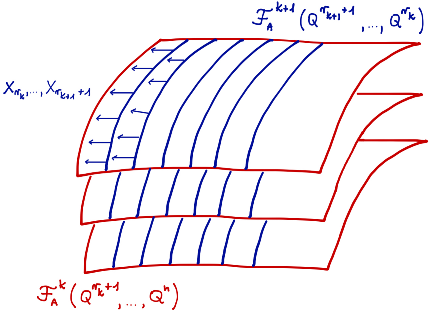
- 

Of course, one understands that if $r_{k+1}=r_k$, then we do not need to choose anything (as $\mathcal{F}^k_A=\mathcal{F}^{k+1}_A$) and we pass to the next step.

Rectification of Abelian algebras of vector fields

- Inductively, suppose $Q^{r_k+1},\dots,Q^n$ are chosen in this way and let $X_{r_{k+1}+1},\dots,X_{r_k}\in L_A^k$ be chosen so that they span a vector subspace of $\mathcal{D}_A^k(p)$ complementary to $\mathcal{D}_A^{k+1}(p)$, i.e.

$$\langle X_{r_{k+1}+1}(p),\dots,X_{r_k}(p)\rangle\oplus\mathcal{D}_A^{k+1}(p)=\mathcal{D}^k(p).$$

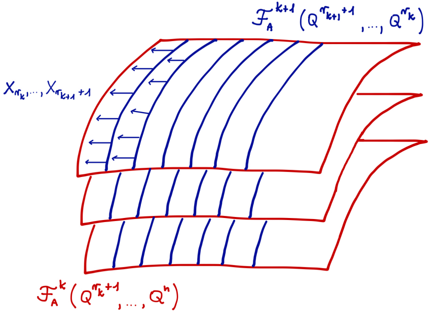
- 

Of course, one understands that if $r_{k+1}=r_k$, then we do not need to choose anything (as $\mathcal{F}_A^k=\mathcal{F}_A^{k+1}$) and we pass to the next step.

Rectification of Abelian algebras of vector fields

- Inductively, suppose $Q^{r_k+1}, \dots, Q^n$ are chosen in this way and let $X_{r_{k+1}+1}, \dots, X_{r_k} \in L_A^k$ be chosen so that they span a vector subspace of $\mathcal{D}_A^k(p)$ complementary to $\mathcal{D}_A^{k+1}(p)$, i.e.

$$\langle X_{r_{k+1}+1}(p), \dots, X_{r_k}(p) \rangle \oplus \mathcal{D}_A^{k+1}(p) = \mathcal{D}^k(p).$$

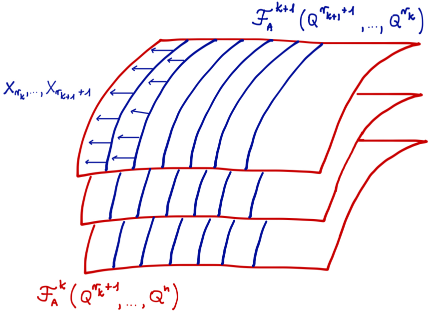
- 

Of course, one understands that if $r_{k+1} = r_k$, then we do not need to choose anything (as $\mathcal{F}_A^k = \mathcal{F}_A^{k+1}$) and we pass to the next step.

Rectification of Abelian algebras of vector fields

- Inductively, suppose $Q^{r_k+1}, \ldots, Q^n$ are chosen in this way and let $X_{r_{k+1}+1}, \ldots, X_{r_k} \in L_A^k$ be chosen so that they span a vector subspace of $\mathcal{D}_A^k(p)$ complementary to $\mathcal{D}_A^{k+1}(p)$, i.e.

$$\langle X_{r_{k+1}+1}(p), \ldots, X_{r_k}(p)\rangle \oplus \mathcal{D}_A^{k+1}(p) = \mathcal{D}^k(p).$$

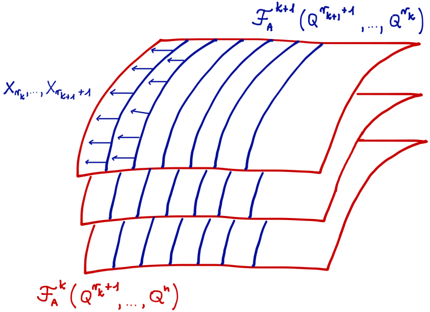
- 

Of course, one understands that if $r_{k+1} = r_k$, then we do not need to choose anything (as $\mathcal{F}_A^k = \mathcal{F}_A^{k+1}$) and we pass to the next step.

Rectification of Abelian algebras of vector fields

- Inductively, suppose $Q^{r_k+1},\ldots,Q^n$ are chosen in this way and let $X_{r_{k+1}+1},\ldots,X_{r_k}\in L^k_A$ be chosen so that they span a vector subspace of $\mathcal{D}^k_A(p)$ complementary to $\mathcal{D}^{k+1}_A(p)$, i.e.

$$\langle X_{r_{k+1}+1}(p),\ldots,X_{r_k}(p)\rangle\oplus\mathcal{D}^{k+1}_A(p)=\mathcal{D}^k(p).$$

- 

Of course, one understands that if $r_{k+1}=r_k$, then we do not need to choose anything (as $\mathcal{F}^k_A=\mathcal{F}^{k+1}_A$) and we pass to the next step.

Rectification of Abelian algebras of vector fields

- On any leaf M^k of the foliation $\mathcal{F}_A^k$ in a neighbourhood of p there are uniquely defined 1-forms $\alpha^{r_{k+1}+1}, \dots, \alpha^{r_k}$ which vanish on $\mathcal{D}_A^{k+1}$ and satisfy

$$\alpha^i(X_j) = \delta_{ij}, \qquad i,j = r_{k+1}+1, \dots, r_k\,.$$

- These forms are closed,

$$\mathrm{d}\alpha^i(X_j, X_l) = X_j(\alpha^i(X_l)) - X_l(\alpha^i(X_j)) - \alpha^i([X_j, X_l]) = 0\,,$$

$i,j,l = r_{k+1}+1, \dots, r_k$,

- so locally of the form $\alpha^i = \mathrm{d}Q^i_{M^k}$ for some (local) functions $Q^i_{M^k}$ on M^k, which can be obtained by quadratures (by integrating α^i's).
- The functions are defined modulo constants, but they could be 'synchronized' along all M^k.

Rectification of Abelian algebras of vector fields

- On any leaf M^k of the foliation $\mathcal{F}_A^k$ in a neighbourhood of p there are uniquely defined 1-forms $\alpha^{r_{k+1}+1}, \ldots, \alpha^{r_k}$ which vanish on $\mathcal{D}_A^{k+1}$ and satisfy

$$\alpha^i(X_j) = \delta_{ij}, \qquad i,j = r_{k+1}+1, \ldots, r_k \,.$$

- These forms are closed,

$$\mathrm{d}\alpha^i(X_j, X_l) = X_j(\alpha^i(X_l)) - X_l(\alpha^i(X_j)) - \alpha^i([X_j, X_l]) = 0\,,$$

$i, j, l = r_{k+1}+1, \ldots, r_k,$

- so locally of the form $\alpha^i = \mathrm{d}Q^i_{M^k}$ for some (local) functions $Q^i_{M^k}$ on M^k, which can be obtained by quadratures (by integrating α^i's).
- The functions are defined modulo constants, but they could be 'synchronized' along all M^k.

Rectification of Abelian algebras of vector fields

- On any leaf M^k of the foliation $\mathcal{F}_A^k$ in a neighbourhood of p there are uniquely defined 1-forms $\alpha^{r_{k+1}+1}, \dots, \alpha^{r_k}$ which vanish on $\mathcal{D}_A^{k+1}$ and satisfy

$$\alpha^i(X_j) = \delta_{ij}, \qquad i, j = r_{k+1} + 1, \dots, r_k \,.$$

- These forms are closed,

$$\mathrm{d}\alpha^i(X_j, X_l) = X_j(\alpha^i(X_l)) - X_l(\alpha^i(X_j)) - \alpha^i([X_j, X_l]) = 0 \,,$$

$i, j, l = r_{k+1} + 1, \dots, r_k$,

- so locally of the form $\alpha^i = \mathrm{d}Q^i_{M^k}$ for some (local) functions $Q^i_{M^k}$ on M^k, which can be obtained by quadratures (by integrating α^i's).
- The functions are defined modulo constants, but they could be 'synchronized' along all M^k.

Rectification of Abelian algebras of vector fields

- On any leaf M^k of the foliation $\mathcal{F}_A^k$ in a neighbourhood of p there are uniquely defined 1-forms $\alpha^{r_{k+1}+1}, \dots, \alpha^{r_k}$ which vanish on $\mathcal{D}_A^{k+1}$ and satisfy

$$\alpha^i(X_j) = \delta_{ij}, \qquad i, j = r_{k+1} + 1, \dots, r_k\,.$$

- These forms are closed,

$$\mathrm{d}\alpha^i(X_j, X_l) = X_j(\alpha^i(X_l)) - X_l(\alpha^i(X_j)) - \alpha^i([X_j, X_l]) = 0\,,$$

$i, j, l = r_{k+1} + 1, \dots, r_k$,

- so locally of the form $\alpha^i = \mathrm{d}Q^i_{M^k}$ for some (local) functions $Q^i_{M^k}$ on M^k, which can be obtained by quadratures (by integrating α^i's).
- The functions are defined modulo constants, but they could be 'synchronized' along all M^k.

Rectification of Abelian algebras of vector fields

- On any leaf M^k of the foliation $\mathcal{F}_A^k$ in a neighbourhood of p there are uniquely defined 1-forms $\alpha^{r_{k+1}+1}, \dots, \alpha^{r_k}$ which vanish on $\mathcal{D}_A^{k+1}$ and satisfy

$$\alpha^i(X_j) = \delta_{ij}, \qquad i, j = r_{k+1}+1, \dots, r_k\,.$$

- These forms are closed,

$$\mathrm{d}\alpha^i(X_j, X_l) = X_j(\alpha^i(X_l)) - X_l(\alpha^i(X_j)) - \alpha^i([X_j, X_l]) = 0\,,$$

$i, j, l = r_{k+1}+1, \dots, r_k$,

- so locally of the form $\alpha^i = \mathrm{d}Q^i_{M^k}$ for some (local) functions $Q^i_{M^k}$ on M^k, which can be obtained by quadratures (by integrating α^i's).
- The functions are defined modulo constants, but they could be 'synchronized' along all M^k.

Rectification of Abelian algebras of vector fields

In this way we get, by quadratures, functions $Q^{r_{k+1}+1}, \ldots, Q^{r_k}$, defined in a neighbourhood U of p, which vanish on $\mathcal{D}_A^{k+1}$. So their level sets, together with those of $Q^{r_k+1}, \ldots, Q^n$, determine (locally) $\mathcal{F}_A^{k+1}$.
Finally, the coordinates have been chosen so that

$$X_j(Q^i) = \alpha^i(X_j) = \delta_{ij}, \qquad i,j = r_{k+1}+1, \ldots, r_k\,,$$

which gives that $L_A^m = A$, imply $X_i = \partial_{Q^i}$ for $i = 1, \ldots, r_m$

Example

Let

$$L = \langle \partial_x, \partial_y, x\partial_x, y\partial_y, y^2\partial_x, y\partial_x \rangle\,.$$

It is a solvable and transitive Lie algebra of vector fields on $\mathbb{R}^2$. The associated descending series is $L^1 = \langle \partial_x, \partial_y, y^2\partial_x, y\partial_x \rangle$,
$A = L^2 = \langle \partial_x, y\partial_x \rangle$, $L^3 = \{0\}$.
A is a rectified Abelian ideal for which $\mathcal{D}_A^2$ is generated by ∂_x.

Rectification of Abelian algebras of vector fields

In this way we get, by quadratures, functions $Q^{r_{k+1}+1}, \dots, Q^{r_k}$, defined in a neighbourhood U of p, which vanish on $\mathcal{D}_A^{k+1}$. So their level sets, together with those of $Q^{r_k+1}, \dots, Q^n$, determine (locally) $\mathcal{F}_A^{k+1}$.

Finally, the coordinates have been chosen so that

$$X_j(Q^i) = \alpha^i(X_j) = \delta_{ij}, \qquad i, j = r_{k+1} + 1, \dots, r_k \,,$$

which gives that $L_A^m = A$, imply $X_i = \partial_{Q^i}$ for $i = 1, \dots, r_m$

Example

Let

$$L = \langle \partial_x, \partial_y, x\partial_x, y\partial_y, y^2\partial_x, y\partial_x \rangle \,.$$

It is a solvable and transitive Lie algebra of vector fields on $\mathbb{R}^2$. The associated descending series is $L^1 = \langle \partial_x, \partial_y, y^2\partial_x, y\partial_x \rangle$, $A = L^2 = \langle \partial_x, y\partial_x \rangle$, $L^3 = \{0\}$.
A is a rectified Abelian ideal for which $\mathcal{D}_A^2$ is generated by ∂_x.

Rectification of Abelian algebras of vector fields

In this way we get, by quadratures, functions $Q^{r_{k+1}+1}, \dots, Q^{r_k}$, defined in a neighbourhood U of p, which vanish on $\mathcal{D}_A^{k+1}$. So their level sets, together with those of $Q^{r_k+1}, \dots, Q^n$, determine (locally) $\mathcal{F}_A^{k+1}$. Finally, the coordinates have been chosen so that

$$X_j(Q^i) = \alpha^i(X_j) = \delta_{ij}, \qquad i,j = r_{k+1}+1, \dots, r_k\,,$$

which gives that $L_A^m = A$, imply $X_i = \partial_{Q^i}$ for $i = 1, \dots, r_m$

Example

Let

$$L = \langle \partial_x, \partial_y, x\partial_x, y\partial_y, y^2\partial_x, y\partial_x \rangle\,.$$

It is a solvable and transitive Lie algebra of vector fields on $\mathbb{R}^2$. The associated descending series is $L^1 = \langle \partial_x, \partial_y, y^2\partial_x, y\partial_x \rangle$, $A = L^2 = \langle \partial_x, y\partial_x \rangle$, $L^3 = \{0\}$.
A is a rectified Abelian ideal for which $\mathcal{D}_A^2$ is generated by ∂_x.

Rectification of Abelian algebras of vector fields

In this way we get, by quadratures, functions $Q^{r_{k+1}+1}, \ldots, Q^{r_k}$, defined in a neighbourhood U of p, which vanish on $\mathcal{D}_A^{k+1}$. So their level sets, together with those of $Q^{r_k+1}, \ldots, Q^n$, determine (locally) $\mathcal{F}_A^{k+1}$. Finally, the coordinates have been chosen so that

$$X_j(Q^i) = \alpha^i(X_j) = \delta_{ij}, \qquad i,j = r_{k+1}+1, \ldots, r_k\,,$$

which gives that $L_A^m = A$, imply $X_i = \partial_{Q^i}$ for $i = 1, \ldots, r_m$

Example

Let

$$L = \langle \partial_x, \partial_y, x\partial_x, y\partial_y, y^2\partial_x, y\partial_x \rangle\,.$$

It is a solvable and transitive Lie algebra of vector fields on $\mathbb{R}^2$. The associated descending series is $L^1 = \langle \partial_x, \partial_y, y^2\partial_x, y\partial_x \rangle$, $A = L^2 = \langle \partial_x, y\partial_x \rangle$, $L^3 = \{0\}$.
A is a rectified Abelian ideal for which $\mathcal{D}_A^2$ is generated by ∂_x.

Rectification of Abelian algebras of vector fields

In this way we get, by quadratures, functions $Q^{r_{k+1}+1}, \dots, Q^{r_k}$, defined in a neighbourhood U of p, which vanish on $\mathcal{D}_A^{k+1}$. So their level sets, together with those of $Q^{r_k+1}, \dots, Q^n$, determine (locally) $\mathcal{F}_A^{k+1}$.
Finally, the coordinates have been chosen so that

$$X_j(Q^i) = \alpha^i(X_j) = \delta_{ij}, \qquad i, j = r_{k+1} + 1, \dots, r_k \,,$$

which gives that $L_A^m = A$, imply $X_i = \partial_{Q^i}$ for $i = 1, \dots, r_m$

Example

Let

$$L = \langle \partial_x, \partial_y, x\partial_x, y\partial_y, y^2\partial_x, y\partial_x \rangle \,.$$

It is a solvable and transitive Lie algebra of vector fields on $\mathbb{R}^2$. The associated descending series is $L^1 = \langle \partial_x, \partial_y, y^2\partial_x, y\partial_x \rangle$, $A = L^2 = \langle \partial_x, y\partial_x \rangle$, $L^3 = \{0\}$.
A is a rectified Abelian ideal for which $\mathcal{D}_A^2$ is generated by ∂_x.

Rectification of Abelian algebras of vector fields

In this way we get, by quadratures, functions $Q^{r_{k+1}+1}, \dots, Q^{r_k}$, defined in a neighbourhood U of p, which vanish on $\mathcal{D}_A^{k+1}$. So their level sets, together with those of $Q^{r_k+1}, \dots, Q^n$, determine (locally) $\mathcal{F}_A^{k+1}$. Finally, the coordinates have been chosen so that

$$X_j(Q^i) = \alpha^i(X_j) = \delta_{ij}, \qquad i, j = r_{k+1} + 1, \dots, r_k \,,$$

which gives that $L_A^m = A$, imply $X_i = \partial_{Q^i}$ for $i = 1, \dots, r_m$

Example

Let

$$L = \langle \partial_x, \partial_y, x\partial_x, y\partial_y, y^2\partial_x, y\partial_x \rangle \,.$$

It is a solvable and transitive Lie algebra of vector fields on $\mathbb{R}^2$. The associated descending series is $L^1 = \langle \partial_x, \partial_y, y^2\partial_x, y\partial_x \rangle$, $A = L^2 = \langle \partial_x, y\partial_x \rangle$, $L^3 = \{0\}$.

A is a rectified Abelian ideal for which $\mathcal{D}_A^2$ is generated by ∂_x.

Rectification of Abelian algebras of vector fields

In this way we get, by quadratures, functions $Q^{r_{k+1}+1},\dots,Q^{r_k}$, defined in a neighbourhood U of p, which vanish on $\mathcal{D}_A^{k+1}$. So their level sets, together with those of $Q^{r_k+1},\dots,Q^n$, determine (locally) $\mathcal{F}_A^{k+1}$.
Finally, the coordinates have been chosen so that

$$X_j(Q^i)=\alpha^i(X_j)=\delta_{ij},\qquad i,j=r_{k+1}+1,\dots,r_k\,,$$

which gives that $L_A^m=A$, imply $X_i=\partial_{Q^i}$ for $i=1,\dots,r_m$

Example

Let

$$L=\langle\partial_x,\partial_y,x\partial_x,y\partial_y,y^2\partial_x,y\partial_x\rangle\,.$$

It is a solvable and transitive Lie algebra of vector fields on $\mathbb{R}^2$. The associated descending series is $L^1=\langle\partial_x,\partial_y,y^2\partial_x,y\partial_x\rangle$,
$A=L^2=\langle\partial_x,y\partial_x\rangle$, $L^3=\{0\}$.
A is a rectified Abelian ideal for which $\mathcal{D}_A^2$ is generated by ∂_x.

Proof of the main theorem

- We shall use induction on the dimension n of the manifold M, but as the considerations are local, we can as well assume that $M = \mathbb{R}^n$.
- The case $n = 0$ is trivial, so assume that $n \geq 1$ and let us pick up an Abelian ideal $A \subset L$ of dimension one or two, whose existence is granted for real solvable finite-dimensional Lie algebras.
- Due to the fact that L is transitive, we know that the distribution $\mathcal{D}_A$ spanned by A is regular, say of rank $r \leq 2$. As it is also involutive, it generates a foliation $\mathcal{F}_A$.
- Moreover, one can obtain by quadratures a coordinate system $Q^1, \dots, Q^n$ such that $\mathcal{D}_A$ is generated by $\partial_{Q^1}, \dots, \partial_{Q^r} \in A$ and leaves of $\mathcal{F}_A$ are the level sets of the functions $Q^{r+1}, \dots, Q^n$.
- We will first consider the case in which the dimension of the Abelian Lie algebra A coincides with the dimension of the integral leaves of the foliation $\mathcal{F}_A$, i.e. $\dim(A) = r$ and $r = 2$ ($r = 1$ goes similarly).

Proof of the main theorem

- We shall use induction on the dimension n of the manifold M, but as the considerations are local, we can as well assume that $M = \mathbb{R}^n$.
- The case $n = 0$ is trivial, so assume that $n \geq 1$ and let us pick up an Abelian ideal $A \subset L$ of dimension one or two, whose existence is granted for real solvable finite-dimensional Lie algebras.
- Due to the fact that L is transitive, we know that the distribution $\mathcal{D}_A$ spanned by A is regular, say of rank $r \leq 2$. As it is also involutive, it generates a foliation $\mathcal{F}_A$.
- Moreover, one can obtain by quadratures a coordinate system $Q^1, \ldots, Q^n$ such that $\mathcal{D}_A$ is generated by $\partial_{Q^1}, \ldots, \partial_{Q^r} \in A$ and leaves of $\mathcal{F}_A$ are the level sets of the functions $Q^{r+1}, \ldots, Q^n$.
- We will first consider the case in which the dimension of the Abelian Lie algebra A coincides with the dimension of the integral leaves of the foliation $\mathcal{F}_A$, i.e. $\dim(A) = r$ and $r = 2$ ($r = 1$ goes similarly).

Proof of the main theorem

- We shall use induction on the dimension n of the manifold M, but as the considerations are local, we can as well assume that $M = \mathbb{R}^n$.
- The case $n = 0$ is trivial, so assume that $n \geq 1$ and let us pick up an Abelian ideal $A \subset L$ of dimension one or two, whose existence is granted for real solvable finite-dimensional Lie algebras.
- Due to the fact that L is transitive, we know that the distribution $\mathcal{D}_A$ spanned by A is regular, say of rank $r \leq 2$. As it is also involutive, it generates a foliation $\mathcal{F}_A$.
- Moreover, one can obtain by quadratures a coordinate system $Q^1, \dots, Q^n$ such that $\mathcal{D}_A$ is generated by $\partial_{Q^1}, \dots, \partial_{Q^r} \in A$ and leaves of $\mathcal{F}_A$ are the level sets of the functions $Q^{r+1}, \dots, Q^n$.
- We will first consider the case in which the dimension of the Abelian Lie algebra A coincides with the dimension of the integral leaves of the foliation $\mathcal{F}_A$, i.e. $\dim(A) = r$ and $r = 2$ ($r = 1$ goes similarly).

Proof of the main theorem

- We shall use induction on the dimension n of the manifold M, but as the considerations are local, we can as well assume that $M = \mathbb{R}^n$.
- The case $n = 0$ is trivial, so assume that $n \geq 1$ and let us pick up an Abelian ideal $A \subset L$ of dimension one or two, whose existence is granted for real solvable finite-dimensional Lie algebras.
- Due to the fact that L is transitive, we know that the distribution $\mathcal{D}_A$ spanned by A is regular, say of rank $r \leq 2$. As it is also involutive, it generates a foliation $\mathcal{F}_A$.
- Moreover, one can obtain by quadratures a coordinate system $Q^1, \dots, Q^n$ such that $\mathcal{D}_A$ is generated by $\partial_{Q^1}, \dots, \partial_{Q^r} \in A$ and leaves of $\mathcal{F}_A$ are the level sets of the functions $Q^{r+1}, \dots, Q^n$.
- We will first consider the case in which the dimension of the Abelian Lie algebra A coincides with the dimension of the integral leaves of the foliation $\mathcal{F}_A$, i.e. $\dim(A) = r$ and $r = 2$ ($r = 1$ goes similarly).

Proof of the main theorem

- We shall use induction on the dimension n of the manifold M, but as the considerations are local, we can as well assume that $M = \mathbb{R}^n$.
- The case $n = 0$ is trivial, so assume that $n \geq 1$ and let us pick up an Abelian ideal $A \subset L$ of dimension one or two, whose existence is granted for real solvable finite-dimensional Lie algebras.
- Due to the fact that L is transitive, we know that the distribution $\mathcal{D}_A$ spanned by A is regular, say of rank $r \leq 2$. As it is also involutive, it generates a foliation $\mathcal{F}_A$.
- Moreover, one can obtain by quadratures a coordinate system $Q^1, \dots, Q^n$ such that $\mathcal{D}_A$ is generated by $\partial_{Q^1}, \dots, \partial_{Q^r} \in A$ and leaves of $\mathcal{F}_A$ are the level sets of the functions $Q^{r+1}, \dots, Q^n$.
- We will first consider the case in which the dimension of the Abelian Lie algebra A coincides with the dimension of the integral leaves of the foliation $\mathcal{F}_A$, i.e. $\dim(A) = r$ and $r = 2$ ($r = 1$ goes similarly).

Proof of the main theorem

- We shall use induction on the dimension n of the manifold M, but as the considerations are local, we can as well assume that $M = \mathbb{R}^n$.
- The case $n = 0$ is trivial, so assume that $n \geq 1$ and let us pick up an Abelian ideal $A \subset L$ of dimension one or two, whose existence is granted for real solvable finite-dimensional Lie algebras.
- Due to the fact that L is transitive, we know that the distribution $\mathcal{D}_A$ spanned by A is regular, say of rank $r \leq 2$. As it is also involutive, it generates a foliation $\mathcal{F}_A$.
- Moreover, one can obtain by quadratures a coordinate system $Q^1, \dots, Q^n$ such that $\mathcal{D}_A$ is generated by $\partial_{Q^1}, \dots, \partial_{Q^r} \in A$ and leaves of $\mathcal{F}_A$ are the level sets of the functions $Q^{r+1}, \dots, Q^n$.
- We will first consider the case in which the dimension of the Abelian Lie algebra A coincides with the dimension of the integral leaves of the foliation $\mathcal{F}_A$, i.e. $\dim(A) = r$ and $r = 2$ ($r = 1$ goes similarly).

Proof of the main theorem

For $\Gamma \in L$, we first conclude that $[\partial_{Q^i}, \Gamma] = \sum_{j=1}^{2} H_i^j \partial_{Q^j}$ implies that

$$\Gamma = \sum_{j=1}^{2} \left(\sum_{i=1}^{2} H_i^j Q^i + b^j(Q^3, \ldots, Q^n) \right) \partial_{Q^j} + \bar{\Gamma}\,,$$

where $H_i^j \in \mathbb{R}$, and b^j as well as the vector field $\bar{\Gamma} = \sum_{s=3}^{n} \gamma^s(Q^3, \ldots, Q^n)\partial_{Q^s}$ depend on coordinates $Q^3, \ldots, Q^n$ only. This leads to a system which in coordinates reads

$$\dot{Q}^j = \sum_{i=1}^{2} H_i^j Q^i + b^j(Q^3, \ldots, Q^n)\,, \quad j = 1, 2\,, \tag{4}$$

$$\dot{Q}^s = \gamma^s(Q^3, \ldots, Q^n)\,, \quad s = 3, \ldots, n\,. \tag{5}$$

Solving (5) by the inductive assumption, we end up with

$$\dot{Q}^j = \sum_{i=1}^{2} H_i^j Q^i + b^j(Q^3(t), \ldots, Q^n(t))\,, \quad j = 1, 2\,,$$

which can be integrated by quadratures.

Proof of the main theorem

For $\Gamma \in L$, we first conclude that $[\partial_{Q^i}, \Gamma] = \sum_{j=1}^{2} H_i^j \partial_{Q^j}$ implies that

$$\Gamma = \sum_{j=1}^{2} \left(\sum_{i=1}^{2} H_i^j Q^i + b^j(Q^3, \dots, Q^n) \right) \partial_{Q^j} + \bar{\Gamma} \,,$$

where $H_i^j \in \mathbb{R}$, and b^j as well as the vector field $\bar{\Gamma} = \sum_{s=3}^{n} \gamma^s(Q^3, \dots, Q^n) \partial_{Q^s}$ depend on coordinates $Q^3, \dots, Q^n$ only. This leads to a system which in coordinates reads

$$\dot{Q}^j = \sum_{i=1}^{2} H_i^j Q^i + b^j(Q^3, \dots, Q^n) \,, \quad j = 1, 2 \,, \tag{4}$$

$$\dot{Q}^s = \gamma^s(Q^3, \dots, Q^n) \,, \quad s = 3, \dots, n \,. \tag{5}$$

Solving (5) by the inductive assumption, we end up with

$$\dot{Q}^j = \sum_{i=1}^{2} H_i^j Q^i + b^j(Q^3(t), \dots, Q^n(t)) \,, \quad j = 1, 2 \,,$$

which can be integrated by quadratures.

Proof of the main theorem

For $\Gamma \in L$, we first conclude that $[\partial_{Q^i}, \Gamma] = \sum_{j=1}^{2} H_i^j \partial_{Q^j}$ implies that

$$\Gamma = \sum_{j=1}^{2} \left(\sum_{i=1}^{2} H_i^j Q^i + b^j(Q^3, \ldots, Q^n) \right) \partial_{Q^j} + \bar{\Gamma}\,,$$

where $H_i^j \in \mathbb{R}$, and b^j as well as the vector field $\bar{\Gamma} = \sum_{s=3}^{n} \gamma^s(Q^3, \ldots, Q^n)\partial_{Q^s}$ depend on coordinates $Q^3, \ldots, Q^n$ only.

This leads to a system which in coordinates reads

$$\dot{Q}^j = \sum_{i=1}^{2} H_i^j Q^i + b^j(Q^3, \ldots, Q^n)\,, \quad j = 1, 2\,, \tag{4}$$

$$\dot{Q}^s = \gamma^s(Q^3, \ldots, Q^n)\,, \quad s = 3, \ldots, n\,. \tag{5}$$

Solving (5) by the inductive assumption, we end up with

$$\dot{Q}^j = \sum_{i=1}^{2} H_i^j Q^i + b^j(Q^3(t), \ldots, Q^n(t))\,, \quad j = 1, 2\,,$$

which can be integrated by quadratures.

Proof of the main theorem

For $\Gamma \in L$, we first conclude that $[\partial_{Q^i}, \Gamma] = \sum_{j=1}^{2} H_i^j \partial_{Q^j}$ implies that

$$\Gamma = \sum_{j=1}^{2} \left(\sum_{i=1}^{2} H_i^j Q^i + b^j(Q^3, \ldots, Q^n) \right) \partial_{Q^j} + \bar{\Gamma} \,,$$

where $H_i^j \in \mathbb{R}$, and b^j as well as the vector field $\bar{\Gamma} = \sum_{s=3}^{n} \gamma^s(Q^3, \ldots, Q^n)\partial_{Q^s}$ depend on coordinates $Q^3, \ldots, Q^n$ only. This leads to a system which in coordinates reads

$$\dot{Q}^j = \sum_{i=1}^{2} H_i^j Q^i + b^j(Q^3, \ldots, Q^n) \,, \quad j = 1, 2 \,, \qquad (4)$$

$$\dot{Q}^s = \gamma^s(Q^3, \ldots, Q^n) \,, \quad s = 3, \ldots, n \,. \qquad (5)$$

Solving (5) by the inductive assumption, we end up with

$$\dot{Q}^j = \sum_{i=1}^{2} H_i^j Q^i + b^j(Q^3(t), \ldots, Q^n(t)) \,, \quad j = 1, 2 \,,$$

which can be integrated by quadratures.

Proof of the main theorem

For $\Gamma \in L$, we first conclude that $[\partial_{Q^i}, \Gamma] = \sum_{j=1}^{2} H_i^j \partial_{Q^j}$ implies that

$$\Gamma = \sum_{j=1}^{2} \left(\sum_{i=1}^{2} H_i^j Q^i + b^j(Q^3, \ldots, Q^n) \right) \partial_{Q^j} + \bar{\Gamma} \,,$$

where $H_i^j \in \mathbb{R}$, and b^j as well as the vector field $\bar{\Gamma} = \sum_{s=3}^{n} \gamma^s(Q^3, \ldots, Q^n)\partial_{Q^s}$ depend on coordinates $Q^3, \ldots, Q^n$ only. This leads to a system which in coordinates reads

$$\dot{Q}^j = \sum_{i=1}^{2} H_i^j Q^i + b^j(Q^3, \ldots, Q^n) \,, \quad j = 1, 2 \,, \tag{4}$$

$$\dot{Q}^s = \gamma^s(Q^3, \ldots, Q^n) \,, \quad s = 3, \ldots, n \,. \tag{5}$$

Solving (5) by the inductive assumption, we end up with

$$\dot{Q}^j = \sum_{i=1}^{2} H_i^j Q^i + b^j(Q^3(t), \ldots, Q^n(t)) \,, \quad j = 1, 2 \,,$$

which can be integrated by quadratures.

Proof of the main theorem

- Now, we should still consider the possibility that the dimension of A is two, but the dimension of the integral leaves of the foliation $\mathcal{D}_A$ is one.
- In this case, we chose a one-dimensional subspace $A_1 \subset A$, whose generator X_1 spans $\mathcal{D}_A$. As we already know, X_1 can be integrated by quadratures and can be taken as ∂_{Q^1} in our system of coordinates.
- From $[\partial_{Q^1}, \Gamma] \in \mathcal{F}_A$ we get that Γ must be of the form

$$\Gamma = \left(f(Q^2,\ldots,Q^n)Q^1 + w(Q^2,\ldots,Q^n)\right)\partial_{Q^1} + \sum_{s=2}^{n}\gamma_s(Q^2,\ldots,Q^n)\partial_{Q^s}\,.$$

- We can first solve, by inductive assumption,

$$\dot{Q}^s = \gamma_s(Q^2,\ldots,Q^n)\,,\quad s = 2,\ldots,n\,,$$

and so reduce to

$$\dot{Q}^1 = f(t)Q^1 + w(t)$$

which also can be solved by quadratures.

Proof of the main theorem

- Now, we should still consider the possibility that the dimension of A is two, but the dimension of the integral leaves of the foliation $\mathcal{D}_A$ is one.
- In this case, we chose a one-dimensional subspace $A_1 \subset A$, whose generator X_1 spans $\mathcal{D}_A$. As we already know, X_1 can be integrated by quadratures and can be taken as ∂_{Q^1} in our system of coordinates.
- From $[\partial_{Q^1}, \Gamma] \in \mathcal{F}_A$ we get that Γ must be of the form

$$\Gamma = \left(f(Q^2, \dots, Q^n)Q^1 + w(Q^2, \dots, Q^n)\right) \partial_{Q^1} + \sum_{s=2}^{n} \gamma_s(Q^2, \dots, Q^n)\partial_{Q^s} .$$

- We can first solve, by inductive assumption,

$$\dot{Q}^s = \gamma_s(Q^2, \dots, Q^n), \quad s = 2, \dots, n,$$

and so reduce to

$$\dot{Q}^1 = f(t)Q^1 + w(t)$$

which also can be solved by quadratures.

Proof of the main theorem

- Now, we should still consider the possibility that the dimension of A is two, but the dimension of the integral leaves of the foliation $\mathcal{D}_A$ is one.
- In this case, we chose a one-dimensional subspace $A_1 \subset A$, whose generator X_1 spans $\mathcal{D}_A$. As we already know, X_1 can be integrated by quadratures and can be taken as ∂_{Q^1} in our system of coordinates.
- From $[\partial_{Q^1}, \Gamma] \in \mathcal{F}_A$ we get that Γ must be of the form

$$\Gamma = \left(f(Q^2, \ldots, Q^n)Q^1 + w(Q^2, \ldots, Q^n)\right)\partial_{Q^1} + \sum_{s=2}^{n} \gamma_s(Q^2, \ldots, Q^n)\partial_{Q^s}.$$

- We can first solve, by inductive assumption,

$$\dot{Q}^s = \gamma_s(Q^2, \ldots, Q^n), \quad s = 2, \ldots, n,$$

and so reduce to

$$\dot{Q}^1 = f(t)Q^1 + w(t)$$

which also can be solved by quadratures.

Proof of the main theorem

- Now, we should still consider the possibility that the dimension of A is two, but the dimension of the integral leaves of the foliation $\mathcal{D}_A$ is one.
- In this case, we chose a one-dimensional subspace $A_1 \subset A$, whose generator X_1 spans $\mathcal{D}_A$. As we already know, X_1 can be integrated by quadratures and can be taken as ∂_{Q^1} in our system of coordinates.
- From $[\partial_{Q^1}, \Gamma] \in \mathcal{F}_A$ we get that Γ must be of the form

$$\Gamma = \left(f(Q^2, \ldots, Q^n)Q^1 + w(Q^2, \ldots, Q^n)\right)\partial_{Q^1} + \sum_{s=2}^{n} \gamma_s(Q^2, \ldots, Q^n)\partial_{Q^s}.$$

- We can first solve, by inductive assumption,

$$\dot{Q}^s = \gamma_s(Q^2, \ldots, Q^n), \quad s = 2, \ldots, n,$$

and so reduce to

$$\dot{Q}^1 = f(t)Q^1 + w(t)$$

which also can be solved by quadratures.

Proof of the main theorem

- Now, we should still consider the possibility that the dimension of A is two, but the dimension of the integral leaves of the foliation $\mathcal{D}_A$ is one.
- In this case, we chose a one-dimensional subspace $A_1 \subset A$, whose generator X_1 spans $\mathcal{D}_A$. As we already know, X_1 can be integrated by quadratures and can be taken as ∂_{Q^1} in our system of coordinates.
- From $[\partial_{Q^1}, \Gamma] \in \mathcal{F}_A$ we get that Γ must be of the form

$$\Gamma = \left(f(Q^2,\dots,Q^n)Q^1 + w(Q^2,\dots,Q^n)\right)\partial_{Q^1} + \sum_{s=2}^{n} \gamma_s(Q^2,\dots,Q^n)\partial_{Q^s}\,.$$

- We can first solve, by inductive assumption,

$$\dot{Q}^s = \gamma_s(Q^2,\dots,Q^n)\,,\quad s = 2,\dots,n\,,$$

and so reduce to

$$\dot{Q}^1 = f(t)Q^1 + w(t)$$

which also can be solved by quadratures.

Proof of the main theorem

- Now, we should still consider the possibility that the dimension of A is two, but the dimension of the integral leaves of the foliation $\mathcal{D}_A$ is one.
- In this case, we chose a one-dimensional subspace $A_1 \subset A$, whose generator X_1 spans $\mathcal{D}_A$. As we already know, X_1 can be integrated by quadratures and can be taken as ∂_{Q^1} in our system of coordinates.
- From $[\partial_{Q^1}, \Gamma] \in \mathcal{F}_A$ we get that Γ must be of the form

$$\Gamma = \left(f(Q^2, \ldots, Q^n)Q^1 + w(Q^2, \ldots, Q^n)\right)\partial_{Q^1} + \sum_{s=2}^{n} \gamma_s(Q^2, \ldots, Q^n)\partial_{Q^s}\,.$$

- We can first solve, by inductive assumption,

$$\dot{Q}^s = \gamma_s(Q^2, \ldots, Q^n)\,, \quad s = 2, \ldots, n\,,$$

and so reduce to

$$\dot{Q}^1 = f(t)Q^1 + w(t)$$

which also can be solved by quadratures.

Proof of the main theorem

- Now, we should still consider the possibility that the dimension of A is two, but the dimension of the integral leaves of the foliation $\mathcal{D}_A$ is one.
- In this case, we chose a one-dimensional subspace $A_1 \subset A$, whose generator X_1 spans $\mathcal{D}_A$. As we already know, X_1 can be integrated by quadratures and can be taken as ∂_{Q^1} in our system of coordinates.
- From $[\partial_{Q^1}, \Gamma] \in \mathcal{F}_A$ we get that Γ must be of the form

$$\Gamma = \left(f(Q^2, \ldots, Q^n)Q^1 + w(Q^2, \ldots, Q^n)\right)\partial_{Q^1} + \sum_{s=2}^{n} \gamma_s(Q^2, \ldots, Q^n)\partial_{Q^s}\,.$$

- We can first solve, by inductive assumption,

$$\dot{Q}^s = \gamma_s(Q^2, \ldots, Q^n)\,, \quad s = 2, \ldots, n\,,$$

and so reduce to

$$\dot{Q}^1 = f(t)Q^1 + w(t)$$

which also can be solved by quadratures.

Example

- To see our method in action, consider the Lie algebra of vector fields in $\mathbb{R}^2$ spanned by

$$X_1 = \partial_x, \quad X_2 = y\partial_x, \quad J = xy\partial_x + (1+y^2)\partial_y .$$

The Lie algebra L is solvable and $A = \langle \partial_x, y\partial_x \rangle$ is its only non trivial ideal.

- If we take $\Gamma = J$ as the dynamical vector field, we immediately see that the Lie's procedure cannot be applied, as J is not an element of any commutative ideal in L.
- Also the mentioned Kozlov's result is not applicable, since the algebra is not triangular and the vector fields are not independent at every point.
- Take $A_1 = \langle \partial_x \rangle$. The equation for the coordinate x in the fibre is

$$\dot{x} = xy .$$

Example

- To see our method in action, consider the Lie algebra of vector fields in $\mathbb{R}^2$ spanned by

$$X_1 = \partial_x, \quad X_2 = y\partial_x, \quad J = xy\partial_x + (1 + y^2)\partial_y.$$

The Lie algebra L is solvable and $A = \langle \partial_x, y\partial_x \rangle$ is its only non trivial ideal.

- If we take $\Gamma = J$ as the dynamical vector field, we immediately see that the Lie's procedure cannot be applied, as J is not an element of any commutative ideal in L.
- Also the mentioned Kozlov's result is not applicable, since the algebra is not triangular and the vector fields are not independent at every point.
- Take $A_1 = \langle \partial_x \rangle$. The equation for the coordinate x in the fibre is

$$\dot{x} = xy .$$

Example

- To see our method in action, consider the Lie algebra of vector fields in $\mathbb{R}^2$ spanned by

$$X_1 = \partial_x, \quad X_2 = y\partial_x, \quad J = xy\partial_x + (1+y^2)\partial_y.$$

The Lie algebra L is solvable and $A = \langle \partial_x, y\partial_x \rangle$ is its only non trivial ideal.

- If we take $\Gamma = J$ as the dynamical vector field, we immediately see that the Lie's procedure cannot be applied, as J is not an element of any commutative ideal in L.
- Also the mentioned Kozlov's result is not applicable, since the algebra is not triangular and the vector fields are not independent at every point.
- Take $A_1 = \langle \partial_x \rangle$. The equation for the coordinate x in the fibre is

$$\dot{x} = xy\,.$$

Example

- To see our method in action, consider the Lie algebra of vector fields in $\mathbb{R}^2$ spanned by

$$X_1 = \partial_x, \quad X_2 = y\partial_x, \quad J = xy\partial_x + (1+y^2)\partial_y.$$

The Lie algebra L is solvable and $A = \langle \partial_x, y\partial_x \rangle$ is its only non trivial ideal.

- If we take $\Gamma = J$ as the dynamical vector field, we immediately see that the Lie's procedure cannot be applied, as J is not an element of any commutative ideal in L.
- Also the mentioned Kozlov's result is not applicable, since the algebra is not triangular and the vector fields are not independent at every point.
- Take $A_1 = \langle \partial_x \rangle$. The equation for the coordinate x in the fibre is

$$\dot{x} = xy\,.$$

Example

- To see our method in action, consider the Lie algebra of vector fields in $\mathbb{R}^2$ spanned by

$$X_1 = \partial_x, \quad X_2 = y\partial_x, \quad J = xy\partial_x + (1+y^2)\partial_y.$$

The Lie algebra L is solvable and $A = \langle \partial_x, y\partial_x \rangle$ is its only non trivial ideal.

- If we take $\Gamma = J$ as the dynamical vector field, we immediately see that the Lie's procedure cannot be applied, as J is not an element of any commutative ideal in L.
- Also the mentioned Kozlov's result is not applicable, since the algebra is not triangular and the vector fields are not independent at every point.
- Take $A_1 = \langle \partial_x \rangle$. The equation for the coordinate x in the fibre is

$$\dot{x} = xy\,.$$

Example

- To see our method in action, consider the Lie algebra of vector fields in $\mathbb{R}^2$ spanned by

$$X_1 = \partial_x, \quad X_2 = y\partial_x, \quad J = xy\partial_x + (1 + y^2)\partial_y.$$

  The Lie algebra L is solvable and $A = \langle \partial_x, y\partial_x \rangle$ is its only non trivial ideal.
- If we take $\Gamma = J$ as the dynamical vector field, we immediately see that the Lie's procedure cannot be applied, as J is not an element of any commutative ideal in L.
- Also the mentioned Kozlov's result is not applicable, since the algebra is not triangular and the vector fields are not independent at every point.
- Take $A_1 = \langle \partial_x \rangle$. The equation for the coordinate x in the fibre is

$$\dot{x} = xy\,.$$

Example

- The differential equation corresponding to the projection $\bar{\Gamma}$ of the dynamical vector field in coordinate y is

$$\dot{y} = 1 + y^2 .$$

- This can be immediately integrated to give $y(t) = y_0 + \tan t$.
- Substituting into the equation in the fibre, we get

$$\dot{x} = (y_0 + \tan t)x ,$$

whose solution can be expressed as $x(t) = x_0 \exp(y_0 t)/\cos t$.

Example

- The differential equation corresponding to the projection $\bar{\Gamma}$ of the dynamical vector field in coordinate y is

$$\dot{y} = 1 + y^2 .$$

- This can be immediately integrated to give $y(t) = y_0 + \tan t$.
- Substituting into the equation in the fibre, we get

$$\dot{x} = (y_0 + \tan t)x ,$$

whose solution can be expressed as $x(t) = x_0 \exp(y_0 t)/\cos t$.

Example

- The differential equation corresponding to the projection $\bar{\Gamma}$ of the dynamical vector field in coordinate y is

$$\dot{y} = 1 + y^2 .$$

- This can be immediately integrated to give $y(t) = y_0 + \tan t$.
- Substituting into the equation in the fibre, we get

$$\dot{x} = (y_0 + \tan t)x ,$$

whose solution can be expressed as $x(t) = x_0 \exp(y_0 t)/\cos t$.

Example

- The differential equation corresponding to the projection $\bar{\Gamma}$ of the dynamical vector field in coordinate y is

$$\dot{y} = 1 + y^2 \, .$$

- This can be immediately integrated to give $y(t) = y_0 + \tan t$.
- Substituting into the equation in the fibre, we get

$$\dot{x} = (y_0 + \tan t)x \, ,$$

whose solution can be expressed as $x(t) = x_0 \exp(y_0 t)/\cos t$.

Problem

- According to Malcev's theorem, any finite-dimensional Lie algebra is a semidirect product of its solvable radical and a semisimple algebra. The semisimple Lie algebras are classified, while the solvable ones do not allow for full classification.
- In view of the above considerations, the question arises how to characterize finite-dimensional transitive and solvable Lie algebras of vector fields. In particular, can such Lie algebras written in some coordinates as polynomial vector fields?
- For instance, the solvable Lie algebra generated in $\mathbb{R}^2$ by

$$\langle \partial_x, x\partial_x, \partial_y, e^y\partial_x \rangle$$

is equivalent to the polynomial one

$$\langle \partial_x, x\partial_x, y\partial_y, y\partial_x \rangle$$

in the half-plane $y > 0$.

Problem

- According to Malcev's theorem, any finite-dimensional Lie algebra is a semidirect product of its solvable radical and a semisimple algebra. The semisimple Lie algebras are classified, while the solvable ones do not allow for full classification.
- In view of the above considerations, the question arises how to characterize finite-dimensional transitive and solvable Lie algebras of vector fields. In particular, can such Lie algebras written in some coordinates as polynomial vector fields?
- For instance, the solvable Lie algebra generated in $\mathbb{R}^2$ by

$$\langle \partial_x, x\partial_x, \partial_y, e^y\partial_x \rangle$$

is equivalent to the polynomial one

$$\langle \partial_x, x\partial_x, y\partial_y, y\partial_x \rangle$$

in the half-plane $y > 0$.

Problem

- According to Malcev's theorem, any finite-dimensional Lie algebra is a semidirect product of its solvable radical and a semisimple algebra. The semisimple Lie algebras are classified, while the solvable ones do not allow for full classification.
- In view of the above considerations, the question arises how to characterize finite-dimensional transitive and solvable Lie algebras of vector fields. In particular, can such Lie algebras written in some coordinates as polynomial vector fields?
- For instance, the solvable Lie algebra generated in $\mathbb{R}^2$ by

$$\langle \partial_x, x\partial_x, \partial_y, e^y\partial_x \rangle$$

is equivalent to the polynomial one

$$\langle \partial_x, x\partial_x, y\partial_y, y\partial_x \rangle$$

in the half-plane $y > 0$.

Problem

- According to Malcev's theorem, any finite-dimensional Lie algebra is a semidirect product of its solvable radical and a semisimple algebra. The semisimple Lie algebras are classified, while the solvable ones do not allow for full classification.
- In view of the above considerations, the question arises how to characterize finite-dimensional transitive and solvable Lie algebras of vector fields. In particular, can such Lie algebras written in some coordinates as polynomial vector fields?
- For instance, the solvable Lie algebra generated in $\mathbb{R}^2$ by

$$\langle \partial_x, x\partial_x, \partial_y, e^y\partial_x \rangle$$

is equivalent to the polynomial one

$$\langle \partial_x, x\partial_x, y\partial_y, y\partial_x \rangle$$

in the half-plane $y > 0$.

Problem

- According to Malcev's theorem, any finite-dimensional Lie algebra is a semidirect product of its solvable radical and a semisimple algebra. The semisimple Lie algebras are classified, while the solvable ones do not allow for full classification.
- In view of the above considerations, the question arises how to characterize finite-dimensional transitive and solvable Lie algebras of vector fields. In particular, can such Lie algebras written in some coordinates as polynomial vector fields?
- For instance, the solvable Lie algebra generated in $\mathbb{R}^2$ by

$$\langle \partial_x, x\partial_x, \partial_y, e^y\partial_x \rangle$$

is equivalent to the polynomial one

$$\langle \partial_x, x\partial_x, y\partial_y, y\partial_x \rangle$$

in the half-plane $y > 0$.

THANK YOU FOR YOUR ATTENTION!