Harmonic maps and shift-invariant subspaces

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September 2018

Harmonic Maps

Definition (J. Eells and J.H. Sampson 1964)

A map $\varphi : (M, g) \to (N, h)$ of Riemannian manifolds is *harmonic* if it is a critical point of the energy functional

$$\int |d\varphi|^2 d\, vol_M$$

on every compact subdomain of M.

The associated Euler-Lagrange equation is

$$\boxed{\tau_\varphi := \mathrm{trace}_g \nabla d\varphi = 0,}$$

where ∇ is the connection on $T^*M \otimes \varphi^{-1}TN$ induced by the Levi-Civita connections on M and N. The quantity $\tau_\varphi \in C^\infty(\varphi^{-1}TN)$ is called the *tension field* of φ.

Harmonic maps

Some remarkable classes of harmonic maps:

- harmonic maps $S^1 \to N$ are the closed geodesics of N, parameterized by arclength;
- holomorphic and anti-holomorphic maps between Kähler manifolds are harmonic;
- a map $\varphi = (\varphi_1, \ldots, \varphi_n) : M \to \mathbb{R}^n$ is harmonic if and only if each component φ_i is a harmonic function in the usual sense, that is, it satisfies the Laplace's equation $\triangle\varphi_i = 0$;
- an isometric immersion $\varphi : (M, g) \to (N, h)$ is a minimal surface (soap film) if and only if it is harmonic;
- several classes of surfaces (e.g. CMC, CGC, Willmore surfaces) are characterized by the harmonicity of a suitable defined Gauss map;
- non-linear sigma models in the physics of elementary particles.

Harmonic maps from surfaces

For two-dimensional domains, we have the following:

- the energy is conformally invariant for the domain metric, hence we can refer to harmonic maps from a Riemann surface M without specify a representative of the conformal class of metrics associated to the complex structure of M;
- the Euler-Lagrange equation takes the form (in a local complex coordinate z of a Riemann surface M):

$$\boxed{(\varphi^*\nabla^N)_{\frac{\partial}{\partial \bar{z}}}\frac{\partial \varphi}{\partial z} = 0,}$$

that is, $\frac{\partial \varphi}{\partial z}$ is holomorphic with respect to the Kozsul-Malgrange holomorphic structure induced by $\varphi^*\nabla^N$ in $\varphi^*T^{\mathbb{C}}N$.

Harmonic maps from surfaces into Grassmannians

- Interpret a smooth map $\psi : M \to Gr_k(\mathbb{C}^n)$ as a smooth complex subbundle $\underline{\psi}$ of the trivial bundle $\underline{\mathbb{C}}^n := M \times \mathbb{C}^n$.
- Define vector bundle morphisms

$$A'_\psi, A''_\psi : \underline{\psi} \to \underline{\psi}^\perp$$

by

$$A'_\psi(v) = \pi^\perp_\psi(\partial_z v), \quad A''_\psi(v) = \pi^\perp_\psi(\partial_{\bar{z}} v)$$

for each smooth section v of $\underline{\psi}$.

- Equip each subbundle $\underline{\psi}$ of $\underline{\mathbb{C}}^n$ with the connection induced from that of $\underline{\mathbb{C}}^n$ and consider on $\underline{\psi}$ the corresponding Koszul–Malgrange holomorphic structure.

ψ is harmonic if and only if A'_ψ and A''_ψ are holomorphic.

Harmonic maps from surfaces into Grassmannians

- Hence, if we have an harmonic map $\psi : M \to Gr_k(\mathbb{C}^n)$, we can remove the singularities in order to obtain a vector subbundle $G^{(1)}(\psi)$ $\underline{\mathbb{C}}^n$ such that

$$G^{(1)}(\psi) = \mathrm{Im} A'_\psi \quad \text{almost everywhere on } M.$$

- The vector bundle $G^{(1)}(\psi)$ represents a new harmonic maps into a certain Grassmannian. Hence we can proceed recursively in order to obtain a sequence

$$\{G^{(r)}(\psi)\}_{r\in\mathbb{Z}}$$

of harmonic maps.

Harmonic maps from surfaces into the unitary group

Let $\varphi : M \to \mathrm{U}(n)$ be a smooth map, where $U(n)$ is equipped with a bi-invariant metric and M is a Riemman surface.

In local coordinates, we write

$$\varphi^{-1} d\varphi =: A dz + B d\bar{z}.$$

The matrix-valued smooth functions A, B satisfy the Maurer-Cartan equation

$$\boxed{A_{\bar{z}} - B_z = [A, B]} \qquad \text{(integrability)}$$

The smooth map φ is harmonic if and only if

$$\boxed{A_{\bar{z}} + B_z = 0} \qquad \text{(harmonicity)}$$

Harmonic maps from surfaces into the unitary group

K. Uhlenbeck (1989) observed that

$$\begin{cases} A_{\bar z} - B_z = [A, B] \\ A_{\bar z} + B_z = 0 \end{cases} \iff \boxed{(A^\lambda)_{\bar z} - (B^\lambda)_z = [A^\lambda, B^\lambda] \quad \text{for all } \lambda \in S^1,}$$

where $A^\lambda = \frac{1}{2}(1-\lambda^{-1})A$ and $B^\lambda = \frac{1}{2}(1-\lambda)B$.

Then, if φ is harmonic, we can integrate to obtain an *extended solution*, that is, a map $\Phi : S^1 \times M \to \mathrm{U}(n)$ satisfying $\Phi(1,\cdot) = I$, $\varphi = \Phi(-1,\cdot)$ and

$$\Phi(\lambda,\cdot)^{-1} d\Phi(\lambda,\cdot) = A^\lambda dz + B^\lambda d\bar z.$$

We can consider Φ as a map from M into the *loop group*

$$\Omega\,\mathrm{U}(n) = \{\gamma : S^1 \to \mathrm{U}(n) \text{ smooth} : \gamma(1) = I\}.$$

Harmonic maps and shift-invariant subspaces

G. Segal (1989) formulated the harmonicity equations for maps from surfaces into $\mathrm{U}(n)$.

- This model associates to each loop $\gamma : S^1 \to \mathrm{U}(n)$ the subspace of $L^2(S^1, \mathbb{C}^n)$ defined by $W = \gamma H_+$, where is the usual Hardy space of $\mathbb{C}^n$-valued functions, i.e.

$$H_+ = \mathrm{Span}\{\lambda^i e_j : \ i \geq 0, j = 1, \ldots, n\}.$$

- If a closed subspace W of $L^2(S^1, \mathbb{C}^n)$ is of the form $W = \gamma H_+$ for some loop $\gamma : S^1 \to \mathrm{U}(n)$, then W is certainly shift-invariant, i.e. $SW \subseteq W$ where S is the forward shift on $L^2(S^1, \mathbb{C}^n)$:

$$(Sf)(\lambda) = \lambda f(\lambda) \qquad (\lambda \in S^1).$$

Harmonic maps and shift-invariant subspaces

- A smooth map $\Phi : M \to \Omega\,\mathrm{U}(n)$ corresponds to a smooth subbundle W of $M \times L^2(S^1, \mathbb{C}^n)$, whose (shift-invariant) fibre at z is given by

$$W(z) = \Phi(z)H_+.$$

- We have: Φ is an extended solution if and only if

$$\boxed{S\partial_z W(z) \subset W(z), \quad \partial_{\bar{z}} W(z) \subset W(z).}$$

  The first condition means that $S\partial_z f(z)$ is a section of W for every smooth section $f : M \to W$. The second condition is interpreted in a similar way and it is equivalent to the holomorphicity of W.
- By extension we shall call the subbundle W an *extended solution* as well.

Harmonic maps and shift-invariant subspaces

- In face of this connection between harmonic maps and families of shift-invariant subspaces, we intend to make use of operator-theoretic methods in order to deduce new results about harmonic maps.
- We shall focus on two major aspects: finite uniton number and symmetry.

Harmonic maps of finite uniton number

We say that the harmonic map $\varphi : M \to \mathrm{U}(n)$ has *finite uniton number* if it admits an extended solution Φ of the form

$$\Phi(\lambda, z) = \sum_{k=-r}^{s} C_k(z)\lambda^k, \quad C_k : M \to \mathfrak{gl}(n, \mathbb{C}) \text{ smooth.}$$

Theorem (K. Uhlenbeck 1989)

Any harmonic map $\varphi : S^2 \to \mathrm{U}(n)$ has finite uniton number.

Harmonic maps of finite uniton number

- The standard factorization theory of matrix-valued functions on S^1 shows that such functions Φ are essentially polynomial *Blaschke-Potapov products* depending on $z \in M$. More precisely, there exist subbundles $\{\alpha_i\}_{i=1}^m$ of $M \times \mathbb{C}^n$ such that

$$\Phi = \prod_{j=1}^{m} (\pi_{\alpha_j} + \lambda \pi_{\alpha_j}^{\perp}).$$

- For $n > 1$, the factors are not necessarily unique. However, the factors can be chosen such that, for $k \leq m$, the partial products

$$\Phi_k := \prod_{j=1}^{k} (\pi_{\alpha_j} + \lambda \pi_{\alpha_j}^{\perp})$$

are extended solutions as well.

New criterion for finite uniton number

Let $\varphi : M \to \mathrm{U}(n)$ be harmonic with extended solution $W = \Phi\mathcal{H}_+$. Consider the operator T, acting on smooth sections of $\underline{\mathcal{H}_+} := M \times \mathcal{H}_+$, defined by

$$\boxed{T = -\lambda^{-1} A_z^\varphi + D_z^\varphi,}$$

where $A_z^\varphi = \frac{1}{2}\varphi^{-1}\varphi_z$ and $D_z^\varphi = \partial_z + A_z^\varphi$.

Remark

1. *The 1-form $D_z^\varphi dz + D_{\bar z}^\varphi d\bar z$ may be interpreted as a unitary connection giving a covariant derivative on the trivial bundle $\underline{\mathbb{C}^n} := M \times \mathbb{C}^n$.*
2. *The harmonic equation is equivalent to $D_{\bar z}^\varphi A_z^\varphi = A_z^\varphi D_{\bar z}^\varphi$, which means that A_z^φ is a holomorphic endomorphism of $\underline{\mathbb{C}^n}$ w.r.t. $D_{\bar z}^\varphi$.*

New criterion for finite uniton number

Consider also the *i-th osculating bundle* of $W = \Phi\mathcal{H}_+$:

$$W_{(i)} = \partial_z^i W + \partial_z^{i-1} W + \ldots + \partial_z W + W.$$

We have:

$$\begin{aligned} W_{(1)} &= \Phi T(\underline{\mathcal{H}_+}) + W \\ W_{(2)} &= \Phi T^2(\underline{\mathcal{H}_+}) + W_{(1)} \\ W_{(3)} &= \Phi T^3(\underline{\mathcal{H}_+}) + W_{(2)} \\ &\ldots \end{aligned}$$

Remark

The harmonicity of φ implies that each $W_{(i)}$ can be extended to a smooth subbundle of $M \times L^2(S^1, \mathbb{C}^n)$ by filling out zeros.

New criterion for finite uniton number

Theorem

Let $W = \Phi\mathcal{H}_+$ be an extended solution associated to $\varphi : M \to \mathrm{U}(n)$. Then the following are equivalent:

(i) *φ is of finite uniton number;*

(ii) *there exists a smooth map $\hat{\Phi} : S^1 \to \mathrm{U}(n)$ (independent of $z \in M$) such that $W \subset \hat{\Phi}\mathcal{H}_+$;*

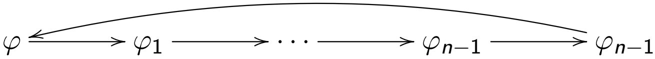
(iii) *the Gauss sequence of W stabilizes, i.e., there exists $i \geq 0$ such that $W_{(i+1)} = W_{(i)}$;*

(iv) *if $T = -\lambda^{-1}A_z^\varphi + D_z^\varphi$, then the maximum power of λ^{-1} in T^r stays bounded when $r \in \mathbb{N}$, i.e., there exists $k_0 \in \mathbb{N}$ such that*

$$\int_0^{2\pi} e^{ikt} T^r(e^{it}) u dt = 0, \quad k \geq k_0,\ u \in C^\infty(M, \mathbb{C}^n),\ r \in \mathbb{N}.$$

New criterion for finite uniton number. Example

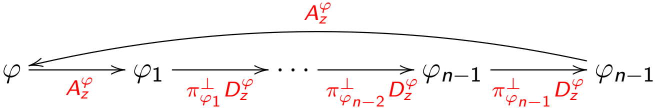
Let $\varphi : M \to \mathbb{C}P^n \hookrightarrow \mathrm{U}(n)$ be a superconformal harmonic map:



Consider again $T = -\lambda^{-1} A_z^\varphi + D_z^\varphi$. A short calculation shows that

$$\begin{array}{rclcrcll} A_z^\varphi(s) & = & -\pi_\varphi^\perp \partial_z s & \text{and} & D_z^\varphi(s) & = & \pi_\varphi \partial_z s & s \in \Gamma(\varphi), \\ A_z^\varphi(s) & = & -\pi_\varphi \partial_z s & \text{and} & D_z^\varphi(s) & = & \pi_\varphi^\perp \partial_z s, & s \in \Gamma(\varphi^\perp). \end{array}$$

Hence



and the maximum power of λ^{-1} in T^{jn} is $2j$.

New criterion for finite uniton number

Corollary

A superconformal harmonic map $\varphi : M \to \mathbb{C}P^{n-1}$ is never of finite uniton number.

In particular, the Clifford torus

$$\mathbb{C} \ni z = x + iy \mapsto (1/\sqrt{2})(\cos 2x, \sin 2x, \cos 2y, \sin 2y) \in S^3.$$

is not of finite uniton number ($S^3 \to \mathbb{R}P^4 \hookrightarrow \mathbb{C}P^4$).

k-symmetric shift-invariant subpaces

Definition

For an integer $k \geq 2$, a shift-invariant subspace W of $L^2(S^1, \mathbb{C}^n)$ is said to be *k-symmetric* if it is invariant with respect to the unitary operator

$$\hat{\omega} : L^2(S^1, \mathbb{C}^n) \to L^2(S^1, \mathbb{C}^n)$$

defined by

$$\hat{\omega}(f)(\lambda) = f(\omega\lambda),$$

where ω is the primitive k-th root of unity.

k-symmetric shift-invariant subpaces

Spectral theorem for $\hat{\omega}|W$:

Theorem

Let W be a k-symmetric shift-invariant subspace and, for $0 \leq j \leq k-1$, set

$$W_j = \{f \in W : \ \hat{\omega}(f) = \omega^j f\}.$$

We have:

(i) $W = \bigoplus_{j=0}^{k-1} W_j$.

(ii) *For $0 \leq j \leq k-1$ there exist closed shift-invariant subspaces V_j of $L^2(S^1, \mathbb{C}^n)$ such that $SV_{k-1} \subseteq V_0 \subseteq V_1 \subseteq \cdots \subseteq V_{k-1}$, and*

$$W_j = S^j\{g \in W : \ g(\lambda) = f(\lambda^k), \ f \in V_j\}.$$

k-symmetric shift-invariant subspaces

Proposition

Let W be a k-symmetric shift-invariant subspace such that $W = \Phi\mathcal{H}_+$ with Φ $\mathrm{U}(n)$-valued a.e. on S^1. Then:

(i) *There exists $\varphi_k \in \mathrm{U}(n)$ with $\varphi_k^k = I$ such that $\Phi(\omega\lambda) = \Phi(\lambda)\varphi_k$.*

(ii) *If $\beta_j = \ker(\varphi_k - \omega^j I)$, and π_j denotes the orthogonal projection from $\mathbb{C}^n$ onto β_j, then*

$$\Phi_k(\lambda) = \Phi(\lambda)\sum_{j=0}^{k-1}\pi_j\lambda^{-j},$$

is a function of λ^k. Set $\Psi(\lambda) = \Phi_k(\lambda^{\frac{1}{k}})$.

(iii) *For $0 \le j \le k-1$ and $\alpha_j = \bigoplus_{l=0}^{j}\beta_l$,*

$$V_j = \Psi(\pi_{\alpha_j} + \lambda\pi_{\alpha_j}^{\perp})\mathcal{H}_+ \qquad W = \Psi(\lambda^k)\sum_{j=0}^{k-1}\pi_j\lambda^j\mathcal{H}_+.$$

k-symmetric extended solutions

We assume throughout that

$$W(z) = \Phi(\cdot, z)\mathcal{H}_+,$$

is k-symmetric for all $z \in M$, with $\Phi : S^1 \times M \to \mathrm{U}(n)$ smooth and $\Phi(1, \cdot) = I$. With the same notations as before, we have

Proposition

The following are equivalent:

(i) W *is an extended solution,*

(ii) $V_0 \subseteq V_1 \subseteq \ldots \subseteq V_{k-1}$ *is a λ-cyclic superhorizontal sequence, that is,* $V_j,\ 0 \leq j \leq k-1$ *are extended solutions,*

$$\partial_z V_j \subseteq V_{j+1},\ 0 \leq j < k-1, \text{ and } \lambda \partial_z V_{k-1} \subseteq V_0.$$

k-symmetric extended solutions

$$\Phi = \Psi(\lambda^k) \sum_{j=0}^{k-1} \pi_j \lambda^j.$$

Proposition

Let $\Psi : S^1 \times M \to \mathrm{U}(n)$ *be an extended solution with* $\Psi(1, \cdot) = I$*, let* $\psi = \Psi(-1, \cdot)$*, and* $A_z^\psi = \frac{1}{2}\psi^{-1}\partial_z\psi$*. If* $\alpha_0 \subseteq \ldots \subseteq \alpha_{k-2}$ *are smooth subbundles of the trivial bundle* $M \times \mathbb{C}^n$*, then* Φ*, with* $\beta_j = \alpha_j \cap \alpha_{j-1}^\perp$*, is an extended solution if and only if the following conditions hold:*

(i) *For* $0 \leq j < k-2$ *we have* $\partial_z \alpha_j \subseteq \alpha_{j+1}$*,*

(ii) $\alpha_{k-2} \subseteq \ker A_z^\psi$ *and* $\mathrm{Im} A_z^\psi \subseteq \alpha_0$*,*

(iii) *For* $0 \leq j \leq k-2$ *we have* $A_z^\psi = \partial_z \pi_{\alpha_j}$ *on* $\alpha_j^\perp$*.*

Primitive harmonic maps

- Given positive integers $r_0, \ldots, r_{k-1}$ with $r_0 + \ldots + r_{k-1} = n$, let $F = F_{r_0,\ldots,r_{k-1}}$ be the flag manifold of ordered sets

$$(A_0, \ldots, A_{k-1})$$

  of complex vector subspaces of $\mathbb{C}^n$, with $\mathbb{C}^n = \bigoplus_{i=0}^{k-1} A_i$ and $\dim A_i = r_i$.
- As a homogeneous space

$$F = \mathrm{U}(n)/\,\mathrm{U}(r_0) \times \ldots \times \mathrm{U}(r_{k-1}).$$

- Fix a point $x_0 = (A_0, \ldots, A_{k-1}) \in F$ and set

$$s(\omega) = \sum_{i=0}^{k-1} \omega^i \pi_{A_i} \in \mathrm{U}(n).$$

  Consider the inner automorphism τ of $\mathfrak{u}(n)$ defined by

$$\tau = s(\omega)^{-1} X s(\omega) \qquad (\tau^k = I).$$

Primitive harmonic maps

- The automorphism τ induces an eigenspace decomposition $\mathfrak{gl}(\mathbb{C}^n) = \sum_{i \in \mathbb{Z}_k} \mathfrak{g}^i$, where

$$\mathfrak{g}^i = \sum_{j \in \mathbb{Z}_k} \mathrm{Hom}(A_j, A_{j-i})$$

  is the ω^i-eigenspace of τ.
- The fixed-set subgroup $\mathrm{U}(n)^\tau$ is precisely the isotropy group at x_0. Hence, F has a canonical structure of *k-symmetric space*.
- F can be embedded in $\mathrm{U}(n)$ as a connected component of $\sqrt[k]{I}$ via the *Cartan embedding* $\iota : F \to \sqrt[k]{I} \subset \mathrm{U}(n)$ defined by $\iota(gx_0) = gs(\omega)g^{-1}$.

Primitive harmonic maps

- Use the Maurer-Cartan form of F to identify

$$T^{\mathbb{C}}F \cong \sum_{i\neq 0}[\mathfrak{g}^i], \qquad [\mathfrak{g}^i]_{gx_0} = g\mathfrak{g}^i g^{-1}.$$

We say that $\varphi : M \to F$ is primitive if $\frac{\partial\varphi}{\partial z}$ is a section of $\varphi^*[\mathfrak{g}^{-1}]$.

$$\varphi(z) = (A_0(z), A_1(z), A_2(z), A_3(z)) \qquad \frac{\partial\varphi}{\partial z} = \begin{pmatrix} 0 & 0 & 0 & * \\ * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \end{pmatrix}$$

Primitive harmonic maps

- If $k \geq 3$, then any primitive map $\varphi : M \to F$ is harmonic with respect to the metric on F induced by the Killing form of $\mathfrak{u}(n)$ (as a matter of fact, φ is harmonic with respect to all invariant metrics on F for which $\mathfrak{g}^{-1}$ is isotropic).
- For $k = 2$, all smooth maps into F are primitive.
- By *primitive harmonic map* into F we mean a primitive map if $k \geq 3$ and a harmonic map if $k = 2$.

Primitive harmonic maps

Let $W = \Phi\mathcal{H}_+$ be a k-symmetric extended solution. Evaluating

$$\boxed{\Phi = \Psi(\lambda^k)\sum_{j=0}^{k-1}\pi_j\lambda^j.}$$

at $\lambda = \omega$, we obtain the map

$$\Phi(\omega) = \sum_{j=0}^{k-1}\pi_j\omega^j,$$

which can be identified via Cartan embedding with the map

$$\varphi = (\beta_0, \beta_1, \ldots, \beta_{k-2}, \beta_{k-1}) : M \to F_{r_0, r_1, \ldots, r_{k-1}},$$

where $r_0 = \dim\beta_0, r_1 = \dim\beta_1, \ldots, r_{k-1} = \dim\beta_{k-1}$.

φ is a primitive harmonic map.

Example

- Let $\psi : M \to \mathbb{C}P^3 \hookrightarrow \mathrm{U}(4)$ be a full holomorphic map and π_ψ be the orthogonal projection onto ψ.
- The corresponding extended solution is $\Psi(\lambda) = \pi_\psi + \lambda \pi_\psi^\perp$ and we have $(A_z^\psi)^2 = 0$.
- Set $\alpha_0 = G^{(1)}(\psi)$ and $\alpha_1 = G^{(1)}(\psi) \oplus G^{(2)}(\psi)$. These subbundles satisfy the conditions of Proposition 3. Then

$$W = (\pi_\psi + \lambda^3 \pi_\psi^\perp)(G^{(1)}(\psi) + \lambda(G^{(1)}(\psi) \oplus G^{(2)}(\psi)) + \lambda^2 \mathcal{H}_+).$$

- The corresponding primitive harmonic map is

$$\varphi : M \to F_{1,1,2}, \qquad \varphi = \left(G^{(1)}(\psi), G^{(2)}(\psi), \psi \oplus G^{(3)}(\psi)\right).$$