Noncommutative gauge theories through twist deformation quantization

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Gel′fand Naimark Theorem (1943)

The notion of a noncommutative space

NCG is based on the correspondence between

Algebra and Geometry

$$\left\{ \begin{array}{l} \textit{compact} \\ \textit{Hausdorff spaces} \end{array} \right\} \simeq \left\{ \begin{array}{l} \textit{commutative unital} \\ C^*\textit{algebras} \end{array} \right\}^{op}$$

$$X \to \mathcal{C}(X)$$

$\mathcal{C}(X)$ = algebra of continuous complex valued functions on X with pointwise multiplication, involution $f \mapsto f^*$, $f^*(x) = \overline{f(x)}$ and $\|f\|_\infty := sup_{x \in M}|f(x)|$.

$$\widehat{A} = \{\chi : A \to \mathbb{C} \text{ character}\} \leftarrow A$$

Motivated by Gelfand-Naimark theorem

$$\left\{ \begin{array}{l} NC\ compact \\ Hausdorff\ spaces \end{array} \right\} := \left\{ \begin{array}{l} (not\ necessarly\ commutative) \\ unital\ C^* - algebras \end{array} \right\}^{op}$$

• While a commutative C^*-algebra has many characters, one for each point of the underlying space, for a noncommutative C^*-algebra characters can be fairly scarce ⇝ NCG is a "point-free" geometry.

• some spaces are better studied by examining algebras of functions on them;
• in part inspired by quantum mechanics:
from commutative algebras of classical observables (= functions on a space)
to noncommutative algebras of quantum observables (= operators on a Hilbert space).

More in general, a NC space is an algebra equipped with some additional structures (C^*, von Neumann, quantum group, spectral triple,....)

Example: NC 2-sphere [Podleś]. $*$-algebra $\mathcal{O}(S^2_q)$ generated by elements $a, a^*, b = b^*$ subject to the relations

$$aa^* + q^{-4}b^2 = 1\,, \quad a^*a + b^2 = 1\,; \quad ab = q^{-2}ba\,, \quad a^*b = q^2ba^* \quad , \quad q \in \mathbb{R}$$

When $q = 1$ one recovers the classical commutative algebra $\mathcal{O}(S^2)$ of polynomials functions on S^2.

NCG has classical geometry (expressed in algebraic terms) as its classical limit.

Bundle theory in noncommutative geometry

Serre-Swan Theorem (1962): the notion of a noncommutative vector bundle

$$\left\{ \begin{array}{l} \textit{vector bundles} \\ \textit{over } X \end{array} \right\} \simeq \left\{ \begin{array}{l} \textit{projective } \mathcal{C}(X) - \textit{modules} \\ \textit{of finite type} \end{array} \right\}$$

$$\begin{array}{rcl} E & \longrightarrow & \mathcal{C}(X)\text{-module } \Gamma(E) \text{ of sections} \\ E_p = \{(x,v) \in X \times \mathbb{C}^N | p(x)v = v\} & \longleftarrow & \mathcal{E}_p \simeq p(\mathcal{C}(X) \otimes \mathbb{C}^N) \end{array}$$

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Remark: Finite projective modules correspond to idempotents in matrix algebras:

$$\begin{array}{rcl} \mathcal{E} \text{ finite projective over } A & \Longleftrightarrow & \exists\, N \in \mathbb{N} \,/\, A \otimes \mathbb{C}^N = \mathcal{E} \oplus \mathcal{E}' \\ & \Longleftrightarrow & \exists\, N \in \mathbb{N}, p = p^2 = p^* \in Mat_N(A) \,/\, \mathcal{E} \simeq p(A \otimes \mathbb{C}^N) \end{array}$$

Example. Podleś 2-sphere S_q^2 with generators $a, a^*, b = b^*$ subject to the relations

$$aa^* + q^{-4}b^2 = 1\,, \quad a^*a + b^2 = 1\,; \quad ab = q^{-2}ba\,, \quad a^*b = q^2 ba^*\,, \quad q \in \mathbb{R}^+$$

The matrix

$$p_q := \frac{1}{2}\begin{pmatrix} 1 + q^{-2}b & q\, a \\ q^{-1}a^* & 1 - b \end{pmatrix} \in Mat(2, S_q^2)$$

is an idempotent $\rightsquigarrow$ vector bundle over S_q^2 (monopole)
(p_q is the Bott projection in the classical limit $q = 1$).

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- notion of equivalence of idempotents $\rightsquigarrow$ K-theory;
- topological invariants (via Connes-Chern pairing)
- purely algebraic def. of differential calculus $(\Omega^n A, d)$ on a (nc) algebra A
- connection on $\mathcal{E}$ is $\nabla : \mathcal{E} \to \mathcal{E} \otimes_A \Omega^1 A$ satisfying Leibniz rule
  (for $\mathcal{E}$ finite projective module, always $\exists$ Grassmann connection: $\nabla := pd$
- ∇^2 curvature $\rightsquigarrow$ gauge theories on NC spaces

Symmetries: from groups to Hopf algebras.

G group of matrices $(G = SL(n,\mathbb{C}),\ SO(n,\mathbb{C}),\dots)$ $\begin{cases} \mu : G \times G \to G,\ (g,h) \mapsto gh \\ e \in G \\ inv : G \to G, g \to g^{-1} \end{cases}$

$\rightsquigarrow$ $\mathcal{O}(G)$ is a Hopf algebra: unital algebra H with

$$\Delta : H \to H \otimes H \text{ coproduct}, \quad h \mapsto h_{(1)} \otimes h_{(2)}$$

$$\varepsilon : H \to \mathbb{C} \text{ counit}$$

$$S : H \to H \text{ antipode}$$

satisfying prop. $1-3$ below.

indeed the group structure induces on $H := \mathcal{O}(G)$ the maps

$$\Delta = \mu^* \quad \varepsilon = ev_e \quad S = inv^*$$

1. μ associative $\Rightarrow (\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$
2. $ge = g = eg \Rightarrow (\varepsilon \otimes id)\Delta = id = (id \otimes \varepsilon)\Delta$
3. $gg^{-1} = e = g^{-1}g \Rightarrow m(S \otimes id)\Delta = \varepsilon(1) = m(id \otimes S)\Delta$

The theory of Hopf algebras has its roots in algebraic topology (Hopf '40s, Sweedler '60s). Later in the 80's: quantum group theory (Faddeev-Reshetikhin-Takhtajan, Drinfeld, Woronowicz, Majid,...).

- Coordinate algebras of quantum groups (FRT bialgebras): $SO_q(n)$, $U_q(n)$, $Sp_q(2n)$,....; $SU_q(2)$ (Woronowicz)
- Quantized universal enveloping algebras (Drinfeld-Jimbo algebras): $\mathcal{U}_q(\mathfrak{g})$

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Definition

Let H be a Hopf algebra, an H-comodule algebra is an algebra A together with an algebra morphism $\delta : A \to A \otimes H$ (coaction) such that

$$(\Delta \otimes id)\delta = (id \otimes \delta)\delta \quad , \quad (\varepsilon \otimes id)\delta = id$$

G-spaces: $\alpha : X \times G \to X$ action dualizes to $\delta := \alpha^* : \mathcal{C}(X) \to \mathcal{C}(X) \otimes \mathcal{C}(G)$

$$x(gh) = (x(g))h \quad \rightsquigarrow \quad (id \otimes \Delta) \circ \delta = (\delta \otimes id) \circ \delta$$

$$(x)e = x \qquad (\mathrm{id} \otimes \varepsilon)\delta = id$$

NC principal bundles & Hopf-Galois extensions [Kreimer, Takeuchi 1981]

- H Hopf algebra (structure group)
- A an H-comodule algebra (total space) with coaction the algebra map

$$\delta : A \to A \otimes H \, , \; a \mapsto a_{(0)} \otimes a_{(1)}$$

- B algebra (base space), $B \simeq A^{co(H)} := \{b \in A | \delta(b) = b \otimes 1_H\}$

+ principality condition: the algebra extension $B \subseteq A$ is Hopf-Galois:

$$\begin{aligned} \chi = (m_A \otimes id)(id \otimes_B \delta) : A \otimes_B A \quad &\to \quad A \otimes H \\ a \otimes_B a' \quad &\mapsto \quad a a'_{(0)} \otimes a'_{(1)} \end{aligned}$$

(canonical map) is bijective.

Example: the 2^{nd} Hopf bundle (instanton bundle)

$$
\begin{array}{ccc}
S^7 \times SU(2) & \xrightarrow{\alpha} & S^7 \\
 & & \downarrow \\
 & & S^4 \simeq S^7/_{SU(2)}
\end{array}
\qquad \leftrightsquigarrow \qquad
\begin{array}{ccc}
A = \mathcal{O}(S^7) & \xleftarrow{\delta = \alpha^*} & \mathcal{O}(S^7) \otimes \mathcal{O}(SU(2)) \\
\uparrow & & \\
B = \mathcal{O}(S^4) \simeq \mathcal{O}(S^7)^{co\,H} & &
\end{array}
$$

principal bundle

Hopf-Galois extension

Example: the 2^{nd} Hopf bundle (instanton bundle)

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\begin{array}{ccc}
S^7 \times SU(2) \xrightarrow{\alpha} S^7 & \rightsquigarrow & A = \mathcal{O}(S^7_q) \xleftarrow{\delta} \mathcal{O}(S^7_q) \otimes \mathcal{O}(SU_q(2)) \\
\downarrow & & \uparrow \\
S^4 \simeq S^7/_{SU(2)} & & B = \mathcal{O}(S^4_q) \simeq \mathcal{O}(S^7_q)^{co\,H}
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principal bundle (family of) Hopf-Galois extensions

Example: the 2^{nd} Hopf bundle (instanton bundle)

$$S^7 \times SU(2) \xrightarrow{\alpha} S^7 \quad\quad A = \mathcal{O}(S^7_q) \xleftarrow{\delta} \mathcal{O}(S^7_q) \otimes \mathcal{O}(SU_q(2))$$

$$S^7 \twoheadrightarrow S^4 \simeq S^7/_{SU(2)} \quad \rightsquigarrow \quad B = \mathcal{O}(S^4_q) \simeq \mathcal{O}(S^7_q)^{co\,H} \hookrightarrow A$$

principal bundle | (family of) Hopf-Galois extensions

- various constructions on different noncommutative spheres, e.g.

  \- from FRT-bialgebras with S^7_q as quantum homog. space of $\mathcal{O}(Sp_q(2))$. Here $q \in \mathbb{R}$, with $q = 1$ classical case. [Landi, P., Reina 2006]

  \- or isospectral deformations $\mathcal{O}(S^m_\Theta)$. Here $\Theta \in Mat(n, \mathbb{R})$, $m = 2n, 2n+1$ antisymmetric, with $\Theta = 0$ classical case. [Landi, Brain, P. , Reina, van Suijlekom,. . . 2005–]
- monopole bundle $S^3 \to S^2$ [Brzezinski, Majid, 1993]
- associated vector bundles $p = p^2$ and (instanton) connections $\nabla = pd$.

Noncommutative principal bundles through twists deformation

[Aschieri, Bieliavsky, P., Schenkel, Commun. Math. Phys 2017]

$$\begin{array}{ccc} A & & \tilde{A} \\ \uparrow H & \overset{deformation\ by\ twists}{\rightsquigarrow} & \uparrow \tilde{H} \\ B = A^{coH} & & \tilde{B} = \tilde{A}^{co\tilde{H}} \end{array}$$

$\longrightarrow$ use the theory of Drinfeld to deform algebra extensions into new algebra extensions in such a way to preserve the condition to be Hopf-Galois, i.e. the invertibility of the canonical map

$$\chi : A \otimes_B A \longrightarrow A \otimes H \quad \in Mor({}_A\mathcal{M}_A^H)$$

$\longrightarrow$ deform (classical or nc) principal bundles into (nc) principal bundles.

Drinfel′d theory of twists

Definition

A linear map $\gamma : H\otimes H \to \mathbb{K}$ is called a **(unital) 2-cocycle** *on H provided*

$$\gamma\big(g_{(1)}\otimes h_{(1)}\big)\gamma\big(g_{(2)}h_{(2)}\otimes k\big)=\gamma\big(h_{(1)}\otimes k_{(1)}\big)\gamma\big(g\otimes h_{(2)}k_{(2)}\big)$$
$$\gamma(h\otimes 1_H)=\varepsilon(h)=\gamma(1_H\otimes h)$$

for all $g,h,k\in H$ (where $h_{(1)}\otimes h_{(2)}=\Delta(h)$ coproduct, sum understood).

Twisting Hopf-algebras:

Let γ be a convolution invertible 2-cocycle on (H,Δ,ε) with inverse $\bar{\gamma}$. Then

$$m_\gamma(h\otimes k):=h\cdot_\gamma k:=\gamma\big(h_{(1)}\otimes k_{(1)}\big)h_{(2)}k_{(2)}\bar{\gamma}\big(h_{(3)}\otimes k_{(3)}\big)$$

defines a new associative product on (the $\mathbb{K}$-module underlying) H.

The resulting algebra $H_\gamma:=(H,m_\gamma,1_H)$ with unchanged coproduct Δ and counit ε and twisted antipode $S_\gamma:=u_\gamma * S * \bar{u}_\gamma$ is a Hopf algebra.

- Deforming spaces carrying H as a symmetry:

$$(A, \delta^A) \in \mathcal{A}^H \quad \rightsquigarrow \quad (A_\gamma, \delta^{A_\gamma}) \in \mathcal{A}^{H_\gamma}$$

If $(A, \delta^A) \in \mathcal{A}^H$ is a right H-comodule algebra with coaction

$$\delta^A : A \to A \otimes H \,, \quad a \mapsto a_{(0)} \otimes a_{(1)}$$

then A, with same coaction, is an H_γ-comodule algebra when endowed with the new product

$$a \otimes a' \mapsto a \bullet_\gamma a' := a_{(0)} a'_{(0)} \bar{\gamma}\Big(a_{(1)} \otimes a'_{(1)}\Big)$$

We denote it by A_γ.

Twisting of Hopf-Galois extensions

Case 1: cocycle $\gamma : H \otimes H \to \mathbb{K}$ on the 'structure group' H

- $H \quad \rightsquigarrow \quad$ twisted Hopf-algebra H_γ
  with twisted product $h \cdot_\gamma k := \gamma\big(h_{(1)} \otimes k_{(1)}\big) h_{(2)} k_{(2)} \bar{\gamma}\big(h_{(3)} \otimes k_{(3)}\big)$
- $A \in \mathcal{A}^H \rightsquigarrow$ twisted comodule-algebra $A_\gamma \in \mathcal{A}^{H_\gamma}$ with same coaction
  and twisted product $a \bullet_\gamma a' := a_{(0)} a'_{(0)} \bar{\gamma}\big(a_{(1)} \otimes a'_{(1)}\big)$
- $B \subseteq A \quad$ is unchanged!
- $\rightsquigarrow$ apply to HG extensions:

$$
\begin{array}{ccc}
A & & A_\gamma \\
H \Big\uparrow & \rightsquigarrow \underset{\gamma \text{ on } H}{\text{twisting}} \rightsquigarrow & H_\gamma \Big\uparrow \\
B = A^{coH} & & B = A_\gamma^{coH_\gamma}
\end{array}
$$

Theorem

The following diagram in ${}_{A_\gamma}\mathcal{M}^{H_\gamma}_{A_\gamma}$ commutes:

$$
\begin{array}{ccc}
A_\gamma \otimes^\gamma_B A_\gamma & \xrightarrow{\ \chi_\gamma\ } & A_\gamma \otimes^\gamma \underline{(H_\gamma)} \\
 & & \Big\downarrow {\scriptstyle \mathrm{id}\otimes^\gamma \mathcal{Q}} \\
\Big\downarrow {\scriptstyle \varphi_{A,A}} & & A_\gamma \otimes^\gamma (\underline{H})_\gamma \\
 & & \Big\downarrow {\scriptstyle \varphi_{A,H}} \\
(A \otimes_B A)_\gamma & \xrightarrow{\ \Gamma(\chi)=\chi\ } & (A \otimes \underline{H})_\gamma
\end{array}
$$

Corollary

The extension $B = A^{coH} \subset A$ is H-Galois $\iff$ the extension $B \simeq A_\gamma^{coH_\gamma} \subset A_\gamma$ is H_γ-Galois.

Case 2: cocycle σ on an external Hopf algebra of symmetries

Let K be a Hopf algebra and σ a 2-cocycle on it.

Suppose that the total space A carries an additional structure of left K-comodule algebra $A \in {}^K\mathcal{A}$ s.t. the coaction $\rho^A : A \to K \otimes A$ is H-equivariant:

$$(\rho^A \otimes id)\delta^A = (\mathrm{id} \otimes \delta^A)\rho^A$$

$$\begin{array}{ccc} A & & {}_\sigma A \\ H \big\uparrow & \rightsquigarrow \underset{\sigma \text{ on } K}{\text{twisting}} \rightsquigarrow & H \big\uparrow \\ B = A^{coH} & & {}_\sigma B \simeq ({}_\sigma A)^{co(H)} \end{array}$$

- ${}_\sigma A$ still carries the coaction of H!
- the base space B is twisted! (while H is unchanged)

Theorem

$B \subseteq A$ is Hopf-Galois if and only if ${}_\sigma B \subseteq {}_\sigma A$ is Hopf-Galois.

EXAMPLE. The quantum Hopf bundle on the Connes-Landi sphere S^4_θ

- Let $K = \mathcal{O}(\mathbb{T}^2)$ be the (commutative) algebra of functions on the 2-torus $\mathbb{T}^2$, $\exists$ a left coaction of $\mathcal{O}(\mathbb{T}^2)$ on the algebra $\mathcal{O}(S^7)$:

$$\rho : \mathcal{O}(S^7) \to \mathcal{O}(\mathbb{T}^2) \otimes \mathcal{O}(S^7)\,, \quad z_i \mapsto \tau_i \otimes z_i$$

which is $\mathcal{O}(SU(2))$-equivariant.

- Let σ be the exponential 2-cocycle on $\mathcal{O}(\mathbb{T}^2)$ determined by setting

$$\sigma\left(t_j \otimes t_k\right) = exp(i\pi\Theta_{jk}); \quad \Theta = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}; \quad \theta \in \mathbb{R}$$

$$\begin{array}{ccc} \mathcal{O}(S^7) & & \mathcal{O}(S^7_\theta) \\ \mathcal{O}(SU(2)) \Big\uparrow & \rightsquigarrow^{\text{twisting}}_{\sigma \text{ on } K} \rightsquigarrow & \Big\uparrow \mathcal{O}(SU(2)) \\ \mathcal{O}(S^4) & & \mathcal{O}(S^4_\theta) \end{array}$$

The resulting bundle is the quantum Hopf bundle on the Connes-Landi sphere $\mathcal{O}(S^4_\theta)$ [Landi, van Suijlekom, 2005].

Remark: Its principality follows from the theory and doesn't need to be proved!

Case 3: combination of deformations

Case 1. γ on H: $(A, H, B) \rightsquigarrow (A_\gamma, H_\gamma, B)$
Case 2. σ on K: $(A, H, B) \rightsquigarrow ({}_\sigma A, H, {}_\sigma B)$

- Let as before A be a right H- comodule algebra with an equivariant left coaction of K
- Let γ a 2-cocycle on H and σ a 2-cocycle on K

$$\begin{array}{ccc} A & & {}_\sigma A_\gamma \\ H \Big\uparrow & \rightsquigarrow \begin{smallmatrix}\text{double}\\ \text{twisting}\end{smallmatrix} \rightsquigarrow & H_\gamma \Big\uparrow \\ B & & {}_\sigma B \end{array}$$

Theorem

$B \subseteq A$ is H-Hopf Galois if and only if ${}_\sigma B \subseteq {}_\sigma A_\gamma$ is H_γ-Hopf Galois.

Application: quantum homogeneous spaces

The gauge group

For a principal G-bundle $\pi : P \to X$, the group $\mathcal{G}_P$ of gauge transformations is

- the subgroup of principal bundle automorphisms which are vertical:

$$\mathcal{G}_P = \mathrm{Aut}_V(P) := \{\varphi : P \to P;\ \varphi(pg) = \varphi(p)g\,, \pi \circ \varphi = \pi\},$$

  with group law given by the composition of maps;

- the group of G-equivariant maps,

$$\mathcal{G}_P = \{\sigma : P \to G;\ \sigma(pg) = g^{-1}\sigma(p)g\}$$

  with pointwise product, $(\sigma \cdot \tau)(p) := \sigma(p)\tau(p) \in G$.

  (Locally, $x \in X \to g(x) \in G$)

The group of gauge transformations acts by pullback on the set $\mathcal{A}_P$ of connections of the bundle $\pi : P \to X$.
ω, η connection forms are gauge equivalent iff $\exists \varphi \in \mathcal{G}_P$ such that $\varphi^* \omega = \eta$.
Indeed gauge equivalence defines an equivalence relation on $\mathcal{A}_P$

$$\rightsquigarrow \mathcal{M} = A_P / \mathcal{G}_P \quad \text{moduli space of connections}$$

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Aim: extend the notion of gauge transformations to the algebraic framework of (NC) Hopf-Galois extensions.

- [Brzeziński (1996)]

Problem: In the classical limit (commutative case) it doesn't give the expected result, but a group bigger than the gauge group of the bundle....

- [Aschieri, Landi, P. (2018)] in the framework of coquasitriangular Hopf algebras

Coquasitriangular Hopf algebras

A Hopf algebra H is called coquasitriangular if it is endowed with a linear map

$$R : H \otimes H \to \mathbb{K} \qquad \text{(universal } r\text{-form)}$$

(with some properties) such that $m_{op} = R * m * \bar{R}$, i.e. for all $h, k \in H$

$$kh = R\big(h_{(1)} \otimes k_{(1)}\big)\, h_{(2)} k_{(2)}\, \bar{R}\big(h_{(3)} \otimes k_{(3)}\big)$$

Examples

- commutative Hopf algebras with trivial universal r-form $R = \varepsilon \otimes \varepsilon$;
- the noncommutative FRT bialgebras $\mathcal{O}_q(G)$ deformations of the algebras of coordinate functions on Lie groups;
- 2-cocycle deformations of coquasitriangular Hopf algebra (H, R) with universal r-form

$$R_\gamma := \gamma_{21} * R * \bar{\gamma} : h \otimes k \longmapsto \gamma\big(k_{(1)} \otimes h_{(1)}\big) R\big(h_{(2)} \otimes k_{(2)}\big) \bar{\gamma}\big(h_{(3)} \otimes k_{(3)}\big)$$

Some useful facts from the theory of cqt Hopf algebras:

• The category $(\mathcal{A}^H, \boxtimes)$ of H-comodule algebras is monoidal: $(A,\delta^A),(C,\delta^C)\in\mathcal{A}^H$, then the H-comodule $A\otimes C$ with tensor product coaction $\delta^{A\otimes C}: a\otimes c\mapsto a_{(0)}\otimes c_{(0)}\otimes a_{(1)}c_{(1)}$ is a right H-comodule algebra,

$$A\boxtimes C := (A\otimes C,\bullet) \quad \text{(braided product algebra)}$$

when endowed with the product

$$(a\otimes c)\bullet(a'\otimes c') := a\, R_{C,A}(c\otimes a')c' = a a'_{(0)}\otimes c_{(0)}c'\, R\big(c_{(1)}\otimes a'_{(1)}\big)\,.$$

• The right H-comodule $\underline{\mathrm{H}} = (H,\mathrm{Ad})$ becomes an H-comodule algebra $\underline{\mathrm{H}} = (H,\star,\mathrm{Ad})$ when endowed with the product

$$h\star k := h_{(2)}k_{(2)}R\big(S(h_{(1)})h_{(3)}\otimes S(k_{(1)})\big)$$

$(\underline{H},\star,\eta,\Delta,\epsilon,\underline{S},\mathrm{Ad})$ is a braided Hopf algebra (associated with H)

Hopf-Galois extensions for coquasitriangular Hopf algebras and their gauge groups.

[P. Aschieri, G. Landi, C.P. (2018)]

Theorem

Let (H,R) be a coquasitriangular Hopf algebra and $A \in \mathcal{A}_{qc}^{(H,R)}$ a quasi-commutative H-comodule algebra. Let $B \subseteq Z(A)$ be the corresponding subalgebra of coinvariants. Then the canonical map

$$\chi = (m \otimes \mathrm{id}) \circ (\mathrm{id} \otimes_B \delta^A) : A \boxtimes_B A \longrightarrow A \boxtimes \underline{H}\,,$$
$$a' \boxtimes_B a \longmapsto a' a_{(0)} \boxtimes a_{(1)}$$

is an algebra map, thus a morphism in $\mathcal{A}^H$.

Definition

Let (H,R) be a coquasitriangular Hopf algebra. A right H-comodule algebra A is quasi-commutative (with respect to the universal r-form R), $A \in \mathcal{A}_{qc}^{(H,R)}$ if

$$m_A = m_A \circ R_{A,A}\,, \qquad ac = c_{(0)} a_{(0)}\, R\big(a_{(1)} \otimes c_{(1)}\big) \qquad a, c \in A$$

Examples

- Clearly, for $(H, \varepsilon \otimes \varepsilon)$, every commutative algebra $A \in \mathcal{A}^H$ is quasi-commutative.
- Twist deformations $A_\gamma \in \mathcal{A}^{H_\gamma}$ of quasi-commutative algebras A via a 2-cocycle on H are quasi-commutative algebras.
- A main example of quasi-commutative comodule algebra is the H-comodule algebra $(\underline{H}, \star, \mathrm{Ad})$ associated with a cotriangular Hopf algebra (H, R).
- $H = \mathcal{O}(GL_q(2))$ is coquasitriangular with (not cotriangular) universal r-form

$$R\Big(u_{ij} \otimes u_{kl}\Big) = q^{-1}\mathcal{R}^{ik}_{jl} \ , \quad R\Big(D^{-1} \otimes u_{ij}\Big) = R\Big(u_{ij} \otimes D^{-1}\Big) = q\,\delta_{ij} \ ,$$

The quantum plane $\mathcal{O}(\mathbb{C}^2_q) = \mathbb{C}[x_, x_2]/\langle x_1 x_2 - q\ x_2 x_1\rangle$ is a quasi-commutative $\mathcal{O}(GL_q(2))$-comodule algebra with coaction $\delta(x_i) = \sum_j x_j \otimes u_{ji}$.

The gauge group of a (coquasi△) Hopf-Galois extension.

Let $B \subseteq A \in \mathcal{A}_{qc}^{(H,R)}$ be an H-Hopf-Galois extension, with H coquasitriangular.

Theorem

The $\mathbb{K}$-module of left B-module, right H-comodule algebra morphisms

$$\mathrm{Aut}_V(A) := \mathrm{Hom}_{{}_B\mathcal{A}^H}(A,A) = \{\mathcal{F} \in \mathrm{Hom}_{\mathcal{A}^H}(A,A), \text{ such that } \mathcal{F}_{|_B} = \mathrm{id}\}$$

is a group with respect to map composition $\mathcal{F} \cdot \mathsf{G} := \mathsf{G} \circ \mathcal{F}$.

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Theorem

The $\mathbb{K}$-module of H-equivariant algebra maps $\underline{H} \to A$

$$\mathcal{G}_A := \mathrm{Hom}_{\mathcal{A}^H}(\underline{H}, A)$$

is a group with respect to the convolution product,
with inverse $\bar{\mathsf{f}} := \mathsf{f} \circ \underline{S}$ for $\mathsf{f} \in \mathrm{Hom}_{\mathcal{A}^H}(\underline{H}, A)$. Moreover, the groups $(\mathcal{G}_A, *)$ and $(\mathrm{Aut}_V(A), \cdot)$ are isomorphic.

Twisting gauge groups

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Let $B = A^{coH} \subseteq A$ be a Hopf-Galois extension and γ a 2-cocycle on H, with H coquasitriangular and $A \in \mathcal{A}_{qc}^{(H,R)}$. The gauge group $\mathcal{G}_{A_\gamma}$ of the twisted Hopf-Galois extension $B = A_\gamma^{coH_\gamma} \subseteq A_\gamma \in \mathcal{A}_{qc}^{(H_\gamma,R_\gamma)}$ is isomorphic to the gauge group $\mathcal{G}_A$ of the initial Hopf-Galois extension.

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Next

- gauge group of a Hopf-Galois extension obtained by twisting for a 2-cocycle on an external Hopf algebra of symmetries K (e.g. instanton bundle on Connes-Landi sphere S^4_θ)

- Gauge group of a generic Hopf-Galois extension? Group or Hopf algebra?