Curvature Effects in Flocking Dynamics: A Cucker-Smale type model on Riemannian Manifolds

joint work with Seung-Yeal Ha and Doheon Kim

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The Cucker-Smale Model

Euclidean case

$$\dot{x}_i = v_i$$
$$\dot{v}_i = \frac{K}{N}\sum_{j=1}^{N} \psi(\|x_i - x_j\|)(v_j - v_i)$$

$N \hat{=}$ number of particles, $K \hat{=}$ coupling strength, $\psi \hat{=}$ communication rate.

The Cucker-Smale Model

Euclidean case

$$\begin{aligned} \dot{x}_i &= v_i \\ \dot{v}_i &= \frac{K}{N}\sum_{j=1}^{N} \psi(\|x_i - x_j\|)(v_j - v_i) \end{aligned}$$

$N \hat{=}$ number of particles, $K \hat{=}$ coupling strength, $\psi \hat{=}$ communication rate.

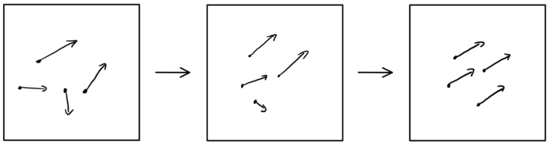

Figure: Example of the time evolution under the Euclidean Cucker-Smale model.

Cucker-Smale model

The Flocking Models

Euclidean Cucker-Smale Model

$$\dot{x}_i = v_i$$

$$\dot{v}_i = \frac{K}{N} \sum_{j=1}^{N} \psi(\|x_i - x_j\|)(v_j - v_i)$$

Riemannian Cucker-Smale Model

$$\dot{x}_i = v_i$$

$$\nabla_{v_i} v_i = \frac{K}{N} \sum_{j=1}^{N} \psi(d_{ij})(P_{ij} v_j - v_i)$$

$N \hat{=}$ number of particles,
$K \hat{=}$ coupling strength,
$\psi \hat{=}$ communication rate,
$\|\cdot\| \hat{=}$ Euclidean norm,

$d_{ij} \hat{=}$ geodesic distance x_j to x_i,
$P_{ij} \hat{=}$ vector transport from x_j to x_i.

One natural choice for P_{ij} is the parallel transport along the distance minimizing geodesic from x_j to x_i. In this case, ψ has to vanishes where this geodesic is not unique.

Immediate Consequences

Riemannian Cucker-Smale Model

$$
\begin{aligned}
\dot{x}_i &= v_i \\
\nabla_{v_i} v_i &= \frac{K}{N} \sum_{j=1}^{N} \psi(d_{ij})(P_{ij} v_j - v_i)
\end{aligned}
$$

- Reduction to the standard Cucker-Smale model for Euclidean space
- Reduction to the Kuramoto model for S^1 with its standard metric en

Results

Flocking Theorem

Given that for all i, j

1. $dist(x_i, x_j)$ is uniformly bounded in time by the injectivity radius of the manifold
2. $\frac{d}{dt}\|P_{ij}v_j - v_i\|^2$ is uniformly bounded in time

then

$$\lim_{t\to\infty}\|P_{ij}v_j - v_i\|^2 = 0 \ \forall i, j$$

Flock Shape Theorem

If the curvature is nowhere vanishing and

$$\lim_{t\to\infty}\|P_{ij}v_j - v_i\|^2 = 0 \ \forall i, j,$$

then all particles have to align along a single geodesic asymptotically.

Some Examples

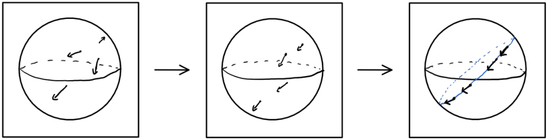

Figure: Example of the time evolution under the Riemannian Cucker-Smale model on S^2.

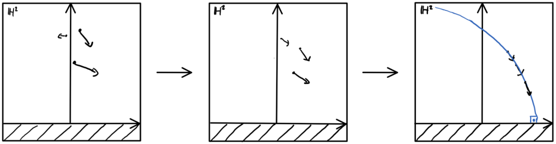

Figure: Example of the time evolution under the Riemannian Cucker-Smale model on the hyperbolic plane (upper half-plane model).

Future Research

- improve on the flocking result
- extend results from the Euclidean case: Mean field limit, coupling to a flow on the manifold, control theory, time delay,...
- generalise geometric input: different constructions of P_{ij}, reformulate from a geometric perspective
- extend study of phenomenology: e.g. topological effects

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Thank you for your attention!

References

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- D. Chi, S.-H. Choi, S.-Y. Ha: *Emergent behaviors of a holonomic particle system on a sphere*. J. Math. Phys. **55:5**, 052703 (2014).