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Abstracts

Session 33

Quantitative Results in Real Algebra and Geometry

Organizers:

Carlos Andradas (Universidad Complutense de Madrid)
Antonio Diaz-Cano (Universidad Complutense de Madrid)
Victoria Powers(Emory University)
Frank Sottile (University of Massachusetts)

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Global semianalytic sets

Francesca Acquistapace (Universita di Pisa)

We present some (few) results that are known for global noncompact semianalytic sets of any dimension. Among these results there is a fake Lojasiewicz inequality that nevertheless allows to get the finiteness property.

Computing the Betti Numbers of Arrangements via Spectral Sequences

Saugata Basu (Georgia Institute of Technology)

In this talk, I will consider the problem of computing the Betti numbers of an arrangement of n compact semi-algebraic sets, $S_1, \ldots, S_n \subset R^k$, where each S_i is described using a constant number of polynomials with degrees bounded by a constant. Such arrangements are ubiquitous in computational geometry. I will describe an algorithm for computing ℓ-th Betti number, $\beta_\ell(\cup_{i=1}^n S_i)$, $0 \leq \ell \leq k-1$, using $O(n^{\ell+2})$ algebraic operations. Additionally, one has to perform linear algebra on integer matrices of size bounded by $O(n^{\ell+2})$. All previous algorithms for computing the Betti numbers of arrangements, triangulated the whole arrangement giving rise to a complex of size $O(n^{2^k})$ in the worst case. Thus, the complexity of computing the Betti numbers (other than the zero-th one) for these algorithms was $O(n^{2^k})$. To our knowledge this is the first algorithm for computing $\beta_\ell(\cup_{i=1}^n S_i)$ that does not rely on such a global triangulation, and has a graded complexity which depends on ℓ.

Convexity Properties of The Cone of Nonnegative Polynomials

Grigoriy Blekherman (University of Michigan)

We study metric properties of the cone of homogeneous non-negative multivariate polynomials and the cone of sums of powers of linear forms, and the relationship between the two cones. We compute the maximum volume ellipsoid of the natural base of the cone of non-negative polynomials and the minimum volume ellipsoid of the natural base of the cone of powers of linear forms and compute the coefficients of symmetry of the bases. The multiplication by $(x_1 2 + \ldots + x_n 2)^m$ induces an isometric embedding of the space of polynomials of degree $2k$ into the space of polynomials of degree $2(k+m)$, which allows us to compare the cone of non-negative polynomials of degree $2k$ and the cone of sums of $2(k+m)$-powers of linear forms. We estimate the volume ratio of the bases of the two cones and the rate at which it approaches 1 as m grows.

The degree of difficulty in avoiding singularities when writing polynomials as sums of squares of real rational functions

Charles N. Delzell (Louisiana State University)

Let $f \in \mathbb{R}[X] := \mathbb{R}[X_1, \ldots, X_n]$ be pos. semidefinite. Call $x \in \mathbb{R}^n$ an *unavoidable singularity* in writing $f = \sum_i r_i^2$ $(r_i \in \mathbb{R}(X))$, if, $\forall$ such $\{r_i\}$, $\exists i$ $r_i(x)$ undefined; i.e., $\forall g, h_i \in \mathbb{R}[X]$, $(g^2 f = \sum_i h_i^2 \;\Rightarrow\; g(x) = 0)$. Let $S(f) = \{\text{such } x\}$ (*Abstracts AMS* **18** (1997), #926-12-174). $r_i(x)$ undefined $\Rightarrow$ r_i has no C^∞ extension at x (Artin approximation); i.e., x is a "singularity."

Question 1: Is $S(f)$ decidable? $\exists g$ $S(f) = Z(g) := \{g(x) = 0\}$. Problem: get g from f effectively.

Question 2: Let $m(f, x) =$ least $m \in \mathbb{N} \cup \{\infty\}$ $[\exists g, h_i \in \mathbb{R}[X]$ with $g^2 f = \sum_i h_i^2$, $\deg g = m$, $g(x) \neq 0]$ (= the "degree of difficulty" above). Can we bound $m(f) := \sup_{x \notin S(f)} m(f, x)$ in $\deg f$? $(S(f) = Z(g) \;\Rightarrow\; m(f) \leq \deg g.)$

Question 3: ¿$[f = \sum_i h_i^2$ $(h_i \in \mathbb{R}[[X]]) \;\Rightarrow\; \mathbf{0} \notin S(f)]$? (Converse obvious.) By Artin, $\forall d \in \mathbb{N}$, $\exists e \in \mathbb{N}$, $\forall f \in \mathbb{R}[X]$ with $\deg f \leq d$: $(\exists g_i \in \mathbb{R}[X]$ with $f \equiv \sum_i g_i^2 \bmod (X)^e) \;\Rightarrow$ $\exists h_i \in \mathbb{R}[[X]]$, alg./$\mathbb{R}(X)$, s.t. $f = \sum_i h_i^2$.

Theorem: For $n = 2$: $S(f) = \emptyset$, but if the coefficients of r_i vary *continuously* in f $(\deg f \leq 6)$, then $\exists f, i$ s.t. r_i has a singularity.

Computational aspects of the Pierce-Birkhoff Conjecture

Laureano González-Vega (Universidad de Cantabria)

The Pierce–Birkhoff Conjecture asks if any piecewise polynomial continuous function on R^d can be written as a finite combination of sup, inf and polynomials in $R[x_1, \ldots, x_d]$.

First, it will be shown how the known solutions of the Pierce–Birkhoff Conjecture for $n = 1$ and $n = 2$ (due to C. Delzell and L. Mahe) convey to new algorithmic procedures to deal with the piecewise polynomial continuous functions defined on R and $R2$. Second, the consideration of piecewise polynomial continuous functions defined on a convex union of d–simplices in R^d (with one polynomial per simplex) allows to derive an algorithm computing the corresponding sup–inf representation for this case.

On the enumerative geometry of real algebraic curves

Johan Huisman (Université Rennes)

Let C be a smooth real plane curve. Let c be its degree and g its genus. We assume that C has at least g real branches, i.e., C is either an M-curve or an $(M-1)$-curve. Let d be a nonzero natural integer strictly less than c. Let e be a partition of cd of length g. Let v be the number of all real plane curves of degree d that are tangent to g real branches of C with orders of tangency the elements of e. We show that v is finite and we determine v explicitly. As an example, let C be a smooth real quintic plane curve. By the genus formula, the genus g of C is equal to 6. Suppose that C is an M-curve, i.e., C has exactly 7 real branches. Then there are 53760 real quartics tangent to 6 real branches of C with orders of tangency $4, 4, 4, 4, 2, 2$.

Ovals of real cyclic p-gonal Riemann surfaces

Milagros Izquierdo (Linköpings Universitet)

A Riemann surface represented by an algebraic equation of the form $y^p = P(x)$, where the coefficients of the polynomial $P(x)$ are real, is a real cyclic p-gonal Riemann surface. The complex conjugation induces a symmetry on this curve. We classify all the symmetries of such a Riemann surface up to conjugacy by means of Fuchsian and NEC groups. The conjugacy classes of symmetries produce non-isomorphic real models of the complex algebraic curve. This is joint work with A. F. Costa.

Representation of Polynomials Positive on Subsets of the Real Line, with Applications to the Multidimensional Moment Problem

Salma Kuhlmann (University of Saskatchewan)

(Joint Work with M. Marshall and Niels Schwartz) We analyse the preorderings and quadratic modules of R[X] associated to semi algebraic subsets of the real line. We give criteria, in terms of the chosen description of the subset, for the preorderings or modules to be closed or saturated. The results are used to investigate the solvability of the Moment Problem in higher dimensions. We provide criteria for the existence of a positive solution to the Moment Problem on subsets of cylinders with compact base. We apply these results to the representation of polynomials positive on non-compact polyhedra. We present open problems concerning the connection between various representations and the Moment Problem.

Sum of squares decompositions for structured polynomials

Pablo A. Parrilo (ETH Zurich)

The decomposition of a multivariate polynomial as a sum of squares is a basic question in real algebraic geometry. It is also one with important consequences, as such decompositions can be used via the Positivstellensatz as easily verifiable certificates of the emptiness of semialgebraic sets. Particularly exciting is the recent availability of efficient techniques, based on convex optimization, for its effective computation. In this talk, we present an overview of the available algorithms, emphasizing our recent developments oriented towards the exploitation of additional algebraic properties, such as symmetries and ideal structure. The ideas and algorithms will be illustrated with examples from a broad range of domains, and the use of the SOSTOOLS software (developed in collaboration with Stephen Prajna and Antonis Papachristodoulou).

Eight points in the plane

Bruce Reznick (University of Illinois)

Given eight points in the plane in general position, there are two linearly independent cubics which pass through them, and which intersect at a ninth point. This fact was used by Hilbert to construct positive semidefinite sextics which are not a sum of squares. We shall discuss this and other aspects of this beautiful and classical situation.

Virtual roots, Budan Fourier theorem, Bernstein basis and root isolation

Marie-Francoise Roy (Université de Rennes I)

The consideration of virtual roots (recently introduced by Gonzalez-Vega, Lombardi and Mahe) gives a new understanding of classical Descartes and Budan-Fourier theorem. The use of the Bernstein basis and De Casteljau's algorithm (typical tools in computer aided geometric design) provides a new view point on Uspensky's method for real root isolation.

Positive polynomials on semialgebraic sets

Niels Schwartz (Universität Passau)

The lecture will present some new results on polynomials that are positive semidefinite on basic closed semialgebraic sets, primarily in the real plane.

Complexity of representations of positive polynomials with applications to optimization

Markus Schweighofer (Université de Rennes I)

Schmüdgen's Positivstellensatz roughly states that a polynomial f positive on a compact basic closed semialgebraic subset S of R^n can be written as a sum of polynomials which are nonnegative on S for certain obvious reasons. However, in general, you have to allow the degree of the summands to exceed largely the degree of f. Phenomena of this type are one of the main problems in the recently popular approximation of nonconvex polynomial optimization problems by semidefinite programs. Prestel proved that there exists a bound on the degree of the summands computable from the following three parameters: The exact description of S, the degree of f and a measure of how close f is to having a zero on S. Roughly speaking, we make explicit the dependence on the second and third parameter. In doing so, the third parameter enters the bound only polynomially.

An algorithm for convexity of semilinear sets over ordered fields

C. Andradas (Universidad Complutense)

R. Rubio (Universidad Antonio de Nebrija)

María Pilar Vélez* (Universidad Antonio de Nebrija)

In this work we study semilinear sets over arbitrary ordered fields. In particular we show that open (resp. closed) semilinear sets are convex if and only if they are basic, that is, a finite intersection of open (resp. closed) halfspaces. As a corollary we get that any bounded convex closed semilinear set is the convex hull of a finite family of points and we obtain a separation result for convex closed semilinear sets.

We recover the constructive proof of these results to get an algorithm for convexity of these sets. This algorithm is polynomial in the number of variables and basic pieces of the initial description; but it is exponential in the number of linear functions which describe the set. An implementation of this method was made in the Computer Algebra System Maple 6. We finish with some examples of this implementation.

Bounds on Betti numbers of semialgebraic sets

Nicolai Vorobjov (University of Bath)

Joint work with A. Gabrielov (Purdue), T. Zell (Purdue-Rennes)

Upper bounds on Betti numbers of semialgebraic sets, in terms of numbers of variables, degrees, and quantity of the defining polynomials, is a classical topic going back to Petrovskii, Oleinik, Thom, and Milnor. Recently the scope of types of formulae for which the singly exponential bounds were known has been significantly extended. The talk will present these new results which cover in particular formulae with alternating quantifiers and with Pfaffian functions.